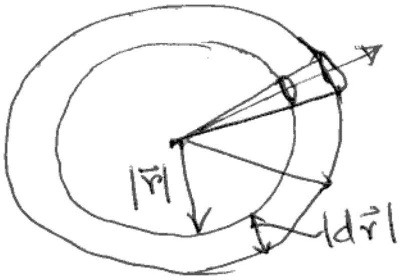
Average Theorem:



Consider the 2 concentric spheres with radii r and $r+dr$.

Average of ϕ over S : $\bar{\phi} = \frac{1}{A}\int \phi \, dA$

where A = area of the sphere $= 4\pi r^2$

& $dA = r^2 d\Omega = r^2 \sin\theta \, d\theta \, d\phi$.

$\therefore \bar{\phi} = \frac{1}{4\pi r^2}\int \phi(\vec{r}) \, r^2 \sin\theta \, d\theta \, d\phi$.

Since r is constant over the sphere, r^2 can be taken out of the integral and we get,

$$\bar{\phi} = \frac{1}{4\pi}\int \phi(\vec{r}) \, d\Omega .$$

Similarly average of ϕ over $\bar{S}$:

$$\bar{\phi}' = \frac{1}{4\pi}\int \phi(\vec{r}+d\vec{r}) \, d\Omega .$$

$\therefore$ Difference, $\bar{\phi}' - \bar{\phi} = \frac{1}{4\pi}\int \left[\phi(\vec{r}+d\vec{r}) - \phi(\vec{r})\right] d\Omega$.

Now, $\phi(\vec{r}+d\vec{r}) = \phi(\vec{r}) + \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz + \cdots$

$$\simeq \phi(\vec{r}) + \vec{\nabla}\phi \cdot d\vec{r}$$

Since, $\vec{\nabla}\phi \cdot d\vec{r} = \left(\hat{x}\frac{\partial \phi}{\partial x} + \hat{y}\frac{\partial \phi}{\partial y} + \hat{z}\frac{\partial \phi}{\partial z}\right)\cdot\left(\hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz\right)$

$$\therefore \bar{\Phi}' - \bar{\Phi} = \frac{1}{4\pi}\int \vec{\nabla}\phi \cdot d\vec{r}\, d\Omega$$

$$= \frac{1}{4\pi r^2}\int \vec{\nabla}\phi \cdot d\vec{r}\, r^2 d\Omega.$$

$$= \frac{1}{4\pi r^2}\int \vec{\nabla}\phi \cdot \hat{r}\, dr\, r^2 d\Omega \qquad (d\vec{r} = \hat{r}\, dr)$$

$$= \frac{dr}{4\pi r^2}\int \vec{\nabla}\phi \cdot \underbrace{\hat{r}\, r^2 d\Omega}_{d\vec{a}}$$

$$= \frac{dr}{4\pi r^2}\int_{\text{surface}} \vec{\nabla}\phi \cdot d\vec{a}$$

Applying divergence theorem,

$$\bar{\Phi}' - \bar{\Phi} = \frac{dr}{4\pi r^2}\int_{\text{Volume}} \vec{\nabla}\cdot(\vec{\nabla}\phi)\, dV = \frac{dr}{4\pi r^2}\int \nabla^2\phi\, dV.$$

$\therefore$ If $\nabla^2\phi = 0 \Rightarrow \bar{\Phi}' - \bar{\Phi} = 0 \Rightarrow \bar{\Phi}' = \bar{\Phi}$

$\therefore$ Average values of ϕ over S & S' are equal.

We can keep doing this by taking smaller and smaller circles until we reach center of sphere.

$\therefore$ If ϕ satisfies Laplace's eqn, the average value of ϕ over a spherical surface equals the value of ϕ at the center of the sphere.