2. Designate the corners of a square, l on a side, in clockwise order, A, B, C, D. Put charges $2q$ at A and $-3q$ at B. Determine the value of the line integral of $\mathbf{E}$, from point C to point D. (No actual integration needed!) What is the numerical answer if $q = 10^{-9}C$ and $l = 5$cm ? [2.5]

Assignment 6 (Sols.)

1. Excess charge, $Q = 5\times10^{8} \times 1.6\times10^{-19}\,C = 8\times10^{-11}\,C$.

Charge density, $\lambda = \frac{Q}{l} = \frac{8\times10^{-11}\,C}{0.04\,m}$

$= 2\times10^{-9}\,C/m$.

(a) Electric field in the rest frame,

$$E = \frac{\lambda}{2\pi\epsilon_0 r} \simeq \frac{2\times10^{-9}\,C/m}{\left(\frac{0.0001}{2}\right)m} \times 18\times10^{9}\,\frac{Nm^2}{C^2}$$

$$\simeq 72\times10^{4}\,V/m. \quad \left(\because \frac{1}{4\pi\epsilon_0} \simeq 9\times10^{9}\,\frac{Nm^2}{C^2}. \quad \therefore \frac{1}{2\pi\epsilon_0} \simeq 18\times10^{9}\,\frac{Nm^2}{C^2}\right)$$

Direction : radial.

(b) In the moving frame,

$$E_r' = \gamma E_r \quad \text{where, } \boxed{\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-(0.9)^2}} \simeq 2.29}$$

$$= 2.29 \times 72\times10^{4}\,V/m$$

$$\simeq 1.65\times10^{6}\,V/m.$$

direction: radial.

PHY102 : Quiz 1

1. A spherical charge distribution has a density ρ that is constant from $r = 0$ out to $r = R$ and is zero beyond. What is the electric field for all values of r, both less than and greater than R ? [2.5]

2. Since the charged particle feels the force due to the electric field only in the y-direction, therefore

$$f_x = \frac{dp_x}{dt} = 0 \implies p_x \text{ is conserved.}$$

However, p is relativistic momentum and is given as $\vec{p} = \gamma m_0 \vec{u}$ where m_0 : rest mass

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$

Since p_x is conserved,

$$\therefore (p_x)_{\text{before entering field}} = (p_x)_{\text{after entering field}}.$$

$$(p_x)_b = \gamma_b m_0 (u_x)_b = \frac{m_0 (u_x)_b}{\sqrt{1 - \frac{(u_x)_b^2}{c^2}}}$$

$$(p_x)_a = \gamma_a m_0 (u_x)_a = \frac{m_0 (u_x)_a}{\sqrt{1 - \frac{(u_x)_a^2 + (u_y)_a^2}{c^2}}}$$

Since, the particle entering the region of the electric field has a y-component of velocity as well.

$$\frac{\cancel{m_0} (u_x)_b}{\sqrt{1 - \frac{(u_x)_b^2}{c^2}}} = \frac{\cancel{m_0} (u_x)_a}{\sqrt{1 - \frac{(u_x)_a^2 + (u_y)_a^2}{c^2}}}$$

$$\implies (u_x)_b^2 \left(1 - \frac{(u_x)_a^2 + (u_y)_a^2}{c^2}\right) = (u_x)_a^2 \left(1 - \frac{(u_x)_b^2}{c^2}\right)$$

PHY102 : Quiz 1

1. A spherical charge distribution has a density ρ that is constant from $r = 0$ out to $r = R$ and is zero beyond. What is the electric field for all values of r, both less than and greater than R ? [2.5]

$$\Rightarrow (u_x)_b^2 - \cancel{\frac{(u_x)_b^2 (u_x)_a^2}{c^2}} - \frac{(u_x)_b^2 (u_y)_a^2}{c^2} = (u_x)_a^2 - \cancel{\frac{(u_x)_a^2 (u_x)_b^2}{c^2}}$$

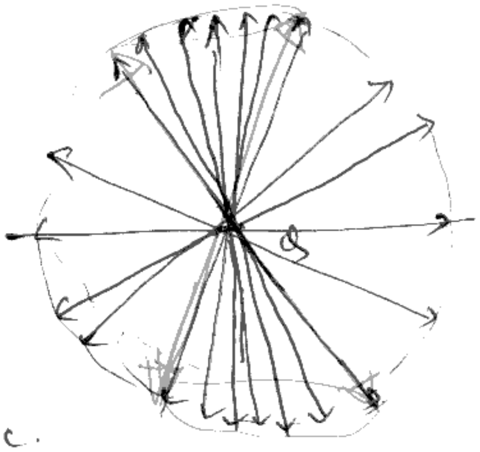
$$\Rightarrow (u_x)_a^2 = (u_x)_b^2 \left[1 - \frac{(u_y)_a^2}{c^2} \right].$$

$$\Rightarrow (u_x)_a = (u_x)_b \sqrt{1 - \frac{(u_y)_a^2}{c^2}}.$$

Since, $(u_y)_a \neq 0$, $\therefore (u_x)_a < (u_x)_b$

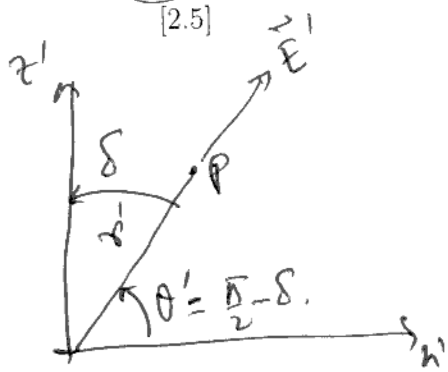
$\therefore$ x component of velocity decreases.

3. The electric field due to a moving charge looks as in the figure – symmetric about a plane $\perp^r$ to direction of charge but not spherically symmetric.



Field stronger at right angles than in direction of motion. This helps understand why half of total flux Q could be contained between 2 conical surfaces as shown in red lines.

2. Designate the corners of a square, l on a side, in clockwise order, A, B, C, D. Put charges $2q$ at A and $-3q$ at B. Determine the value of the line integral of $\mathbf{E}$, from point C to point D. (No actual integration needed!) What is the numerical answer if $q = 10^{-9}C$ and $l = 5\text{cm}$? [2.5]



Now, $$E' = \frac{1}{4\pi\epsilon_0}\frac{Q}{r'^2}\frac{1-\beta^2}{(1-\beta^2\sin^2\theta')^{3/2}}$$

If flux through 2 conical surfaces is half the total flux from Q, then by symmetry,

flux from Q between $\theta' = 0$ & $\frac{\pi}{2}-\delta$ = flux from Q between $\theta' = \frac{\pi}{2}-\delta$ to $\frac{\pi}{2}$.

Now, flux $= E' \cdot 2\pi r'^2 \sin\theta'\, d\theta'$.

$$\therefore \int_0^{\pi/2-\delta} \frac{1}{4\pi\epsilon_0}\frac{Q}{r'^2}\frac{(1-\beta^2)\cdot 2\pi r'^2 \sin\theta'\, d\theta'}{(1-\beta^2\sin^2\theta')^{3/2}}$$

$$= \int_{\pi/2-\delta}^{\pi} \frac{1}{4\pi\epsilon_0}\frac{Q}{r'^2}\frac{(1-\beta^2)2\pi r'^2 \sin\theta'\, d\theta'}{(1-\beta^2\sin^2\theta')^{3/2}}.$$

$$\Rightarrow \int_0^{\pi/2-\delta}\frac{\sin\theta'\, d\theta'}{(1-\beta^2\sin^2\theta')^{3/2}} = \int_{\pi/2-\delta}^{\pi/2}\frac{\sin\theta'\, d\theta'}{(1-\beta^2\sin^2\theta')^{3/2}}.$$

$$\Rightarrow \int_0^{\pi/2-\delta}\frac{\sin\theta'\, d\theta'}{(1-\beta^2+\beta^2\cos^2\theta')^{3/2}} = \int_{\pi/2-\delta}^{\pi/2}\frac{\sin\theta'\, d\theta'}{(1-\beta^2+\beta^2\cos^2\theta')^{3/2}}$$

2

$(\because \sin^2\theta' = 1-\cos^2\theta')$.

PHY102 : Quiz 1

1. A spherical charge distribution has a density ρ that is constant from $r = 0$ out to $r = R$ and is zero beyond. What is the electric field for all values of r, both less than and greater than R ? [2.5]

Let, $p = \beta \cos\theta'$ ∴ $dp = -\beta \sin\theta' \, d\theta'$

θ	0	$\pi/2 - \delta$	$\pi/2$
p	β	$\beta \cos\left(\frac{\pi}{2} - \delta\right)$	0

$$\therefore \int_{\beta}^{\beta\cos\left(\frac{\pi}{2}-\delta\right)} \frac{-dp/\beta}{\left(1-\beta^2+p^2\right)^{3/2}} = \int_{\beta\cos\left(\frac{\pi}{2}-\delta\right)}^{0} \frac{-dp/\beta}{\left(1-\beta^2+p^2\right)^{3/2}}$$

$$\Rightarrow \int_{\beta\cos\left(\frac{\pi}{2}-\delta\right)}^{\beta} \frac{dp}{\left(1-\beta^2+p^2\right)^{3/2}} = \int_{0}^{\beta\cos\left(\frac{\pi}{2}-\delta\right)} \frac{dp}{\left(1-\beta^2+p^2\right)^{3/2}}$$

Since, $\displaystyle\int \frac{dx}{\left(a^2+x^2\right)^{3/2}} = \frac{x}{a^2\left(a^2+x^2\right)^{1/2}}$ (prob. 5.11).

∴ taking, $a^2 = 1-\beta^2$,

$$\left[\frac{p}{\cancel{\left(1-\beta^2\right)}\left(1-\beta^2+p^2\right)^{1/2}}\right]_{\beta\cos\left(\frac{\pi}{2}-\delta\right)}^{\beta} = \left[\frac{p}{\cancel{\left(1-\beta^2\right)}\left(1-\beta^2+p^2\right)^{1/2}}\right]_{0}^{\beta\cos\left(\frac{\pi}{2}-\delta\right)}$$

$$\Rightarrow \beta - \frac{\beta\cos\left(\frac{\pi}{2}-\delta\right)}{\left[1-\beta^2\sin^2\left(\frac{\pi}{2}-\delta\right)\right]^{1/2}} = \frac{\beta\cos\left(\frac{\pi}{2}-\delta\right)}{\left[1-\beta^2\sin^2\left(\frac{\pi}{2}-\delta\right)\right]^{1/2}} - 0.$$

2. Designate the corners of a square, l on a side, in clockwise order, A, B, C, D. Put charges $2q$ at A and $-3q$ at B. Determine the value of the line integral of $\mathbf{E}$, from point C to point D. (No actual integration needed!) What is the numerical answer if $q = 10^{-9}C$ and $l = 5\text{cm}$? [2.5]

$$\therefore \cancel{\beta} = \frac{2\cancel{\beta}\sin\delta}{\left[1-\beta^2\cos^2\delta\right]^{1/2}}$$

$$\Rightarrow 1-\beta^2\cos^2\delta = 4\sin^2\delta$$

$$\Rightarrow 1-\beta^2+\beta^2\sin^2\delta = 4\sin^2\delta$$

$$\Rightarrow (4-\beta^2)\sin^2\delta = 1-\beta^2$$

$$\therefore \sin\delta = \pm\sqrt{\frac{1-\beta^2}{4-\beta^2}}$$

$$= \pm\frac{1}{\gamma}\left(\frac{1}{4-\beta^2}\right)^{1/2} \quad \left(\because \gamma^2 = \frac{1}{1-\beta^2}\right)$$

For $\gamma \gg 1$, δ is small $\therefore \sin\delta \sim \delta$.

Also, $\beta \simeq 1$ for large γ.

$$\therefore \delta \simeq \pm\frac{1}{\gamma}\frac{1}{\sqrt{3}}$$

PHY102 : Quiz 1

1. ~~A spherical charge distribution has a density ρ that is constant from $r = 0$ out to $r = R$ and is zero beyond. What is the electric field for all values of r, both less than and greater than R ?~~ [2.5]

4. In the rest frame of the protons, the electrostatic force between the 2 protons is just $\frac{e^2}{4\pi\epsilon_0 r^2}$.

$$\therefore f_{rest} = \frac{e^2}{4\pi\epsilon_0 r^2}.$$

Now, getting back to the lab frame, which is moving with velocity βc,

$$f = \frac{1}{\gamma} f_{rest} = \frac{1}{\gamma}\frac{e^2}{4\pi\epsilon_0 r^2}.$$

However, at the instantaneous position of one of the protons, the electric field strength caused by the other is $\frac{\gamma e}{4\pi\epsilon_0 r^2}$.

$$\therefore \text{Discrepancy} = \frac{1}{\gamma}\frac{e^2}{4\pi\epsilon_0 r^2} - \left(\frac{\gamma e}{4\pi\epsilon_0 r^2}\right)\cdot e$$

$$= \frac{e^2}{4\pi\epsilon_0 r^2}\left(\frac{1-\gamma^2}{\gamma}\right).$$

$\gamma^2 = \frac{1}{1-\beta^2}$

$\therefore \frac{1}{\gamma^2} = 1-\beta^2$

$$= \frac{e^2}{4\pi\epsilon_0 r^2}\gamma\left(\frac{1}{\gamma^2} - 1\right).$$

$$= -\frac{e^2}{4\pi\epsilon_0 r^2}\gamma\beta^2 = -\frac{e^2}{4\pi\epsilon_0 r^2}\cdot\gamma\frac{v}{c}\beta.$$

$$= -ev\underbrace{\left(\frac{\gamma e}{4\pi\epsilon_0 r^2}\cdot\frac{\beta}{c}\right)}_{B\text{ (magnetic field)}}.$$

$$\therefore B = \frac{\gamma e}{4\pi\epsilon_0 r^2}\frac{\beta}{c} = \frac{\beta}{c}E$$

Note: It is (β/c) times the electric field & not β as in question.

2. ~~Designate the corners of a square, l on a side, in clockwise order, A, B, C, D. Put charges $2q$ at A and $-3q$ at B. Determine the value of the line integral of **E**, from point C to point D. (No actual integration needed!) What is the numerical answer if $q = 10^{-9}C$ and $l = 5$cm ?~~ [2.5]

5. In frame F, $I_k = \lambda_k \beta_k c$.

In the frame F', $\beta_k' = \dfrac{\beta_k + \beta}{1 + \beta_k \beta}$ since F' moves

$\parallel^l$ to line with velocity $(-\beta c)$.

Now, $\lambda_k' = \dfrac{\lambda_k \gamma_k'}{\gamma_k}$ (recall the argument from what I did in the ~~cl~~ class. F $\longrightarrow$ rest frame $\longrightarrow$ F').

$$\gamma_k = \frac{1}{\sqrt{1-\beta_k^2}} \; ; \quad \gamma_k' = \frac{1}{\sqrt{1-\beta_k'^2}} \; , \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

$$\therefore \gamma_k' = \frac{1}{\sqrt{1 - \left(\frac{\beta_k + \beta}{1+\beta_k\beta}\right)^2}} = \gamma\gamma_k(1+\beta\beta_k).$$

$$\therefore I_k' = \lambda_k' \beta_k' c = \lambda_k \cdot \frac{\gamma \cancel{\gamma_k} \cancel{(1+\beta\beta_k)}}{\cancel{\gamma_k}} \left(\frac{\beta_k + \beta}{\cancel{1+\beta\beta_k}}\right) \cdot c$$

$$= \lambda_k \gamma (\beta_k + \beta) c = \gamma(\lambda_k \beta_k c + \lambda_k \beta c)$$

$$= \gamma(I_k + \beta c \lambda_k).$$

$$\lambda_k' = \lambda_k \cdot \frac{\gamma \cancel{\gamma_k}(1+\beta\beta_k)}{\cancel{\gamma_k}} = \gamma\left(\lambda_k + \frac{\beta I_k}{c}\right)$$

Total, $\lambda = \sum_k \lambda_k$ & $I = \sum_k I_k$

$$\therefore \lambda' = \gamma\left(\lambda + \frac{\beta I}{c}\right) \; ; \quad I' = \gamma(I + \beta c \lambda).$$