Assignment 3.

1. (a) $$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y & -x+y & -2z \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial}{\partial y}(-2z) - \frac{\partial}{\partial z}(-x+y)\right) - \hat{y}\left(\frac{\partial}{\partial x}(-2z) - \frac{\partial}{\partial z}(x+y)\right)$$
$$+ \hat{z}\left(\frac{\partial}{\partial x}(-x+y) - \frac{\partial}{\partial y}(x+y)\right)$$

$$= \hat{z}(-1-1) = -2\hat{z} \neq 0$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial y}(-x+y) + \frac{\partial}{\partial z}(-2z)$$

$$= 1+1-2 = 0.$$

Not an $\vec{E}$ field.

(b) $\vec{\nabla}\times\vec{G} = 0$ (Show!)

$\vec{\nabla}\cdot\vec{G} = 0$ (Show!)

~~Not~~ Possible $\vec{E}$ field. $\vec{G}$ can be written down as gradient of some potential ϕ.

Note $\vec{\nabla}\cdot\vec{G} = 0$ does not imply that ~~the~~ $\vec{G} = 0$.

Since $\vec{G}$ is a possible $\vec{E}$ field, $\therefore \vec{\nabla}\cdot\vec{G} = \rho/\epsilon_0 = 0$.

i.e. $\rho = 0$. which is possible as we have seen ~~for a~~ inside a conductor.

As examples, you can look at the vector fields (c) and (d) in Purcell Ch 2. fig 2.30. In both cases, $\vec{\nabla}\times\vec{F} = 0$ & $\vec{\nabla}\cdot\vec{F} = 0$.

Since, $\vec{G} = -\vec{\nabla}\phi \Rightarrow \therefore \phi(\vec{x}) - \phi(0) = \int_0^{\vec{x}} \vec{G}(\vec{x})\cdot d\vec{s}$

Choosing $\phi(0) = 0$ & choosing an arbitrary path such as.

$$(0,0,0) \longrightarrow (x_1, 0, 0) \longrightarrow (x_1, y_1, 0) \longrightarrow (x_1, y_1, z_1)$$

$$\phi(\vec{x}) = \underbrace{\int_{(0,0,0)}^{(x_1,0,0)} \vec{G}\cdot d\vec{s}}_{\substack{\text{Along this path} \\ \cancel{x=x_1}\ y = 0 = z \\ \& \ dy = 0 = dz}} + \underbrace{\int_{(x_1,0,0)}^{(x_1,y_1,0)} \vec{G}\cdot d\vec{s}}_{\substack{x = x_1,\ z = 0 \\ \& \ dx = 0 = dz}} + \underbrace{\int_{(x_1,y_1,0)}^{(x_1,y_1,z_1)} \vec{G}\cdot d\vec{s}}_{\substack{x = x_1,\ y = y_1 \\ \& \ dx = dy = 0.}}$$

$$\therefore \phi(\vec{x}) = 0 + 2x_1 \int_0^{y_1} dy + 3y_1 \int_0^{z_1} dz$$

$$= 2x_1 y_1 + 3y_1 z_1$$

$$\therefore \phi(x,y,z) = 2xy + 3yz. \quad \text{(independent of path).}$$

(c) Show, $\vec{\nabla}\times\vec{H} \neq 0$ & $\vec{\nabla}\cdot\vec{H} \neq 0$.

2. (a) $$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} + \frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial y \partial x} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y}.$$

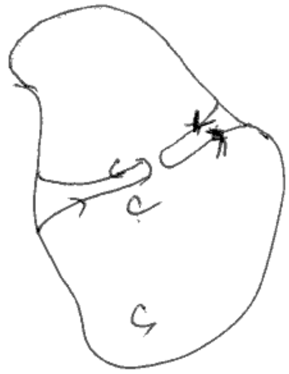
Assuming continuous derivatives, i.e. if

$\frac{\partial}{\partial x_i} P(x_i, x_j, \ldots)$ are continuous for all x_i.

then, $$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} P = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} P.$$

$$\therefore \frac{\partial^2 A_z}{\partial x \partial y} = \frac{\partial^2 A_z}{\partial y \partial x} \quad \& \text{ so on.}$$

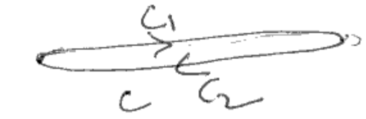
$$\therefore \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0.$$

(b) 

For the closed surface S, consider a closed curve C which leaves an infinitesimal slit.

If $\vec{A}$ is well behaved on S, then

$$\oint_C \vec{A}\cdot d\vec{s} = 0.$$



$\vec{A}\cdot d\vec{s}$ along C_1 = $-\vec{A}\cdot d\vec{s}$ along C_2.

The path breaks closed surface S into open surface S' with a small slit. &

$$\int_{S'} (\vec{\nabla}\times\vec{A})\cdot d\vec{a}' = 0. \quad \text{(Stokes')}$$

Since the slit can be made arbitrarily small, the conclusion holds for the closed surface S,

$$\therefore \int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{a} = 0.$$

Take $\vec{F} = \vec{\nabla}\times\vec{A}$. Then, by divergence theorem,

$$\int_S \vec{F}\cdot d\vec{a} = \oint_V (\vec{\nabla}\cdot\vec{F})\, dv$$

$$\Rightarrow \int_S (\vec{\nabla}\times\vec{F})\cdot d\vec{a} = \oint_V \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})\, dv = 0.$$

Since this is true for any arbitrary volume,

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0.$$

3. $\phi = \phi_0 e^{-kz}\cos kx$.

(a) $$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad \text{(Show!)}$$

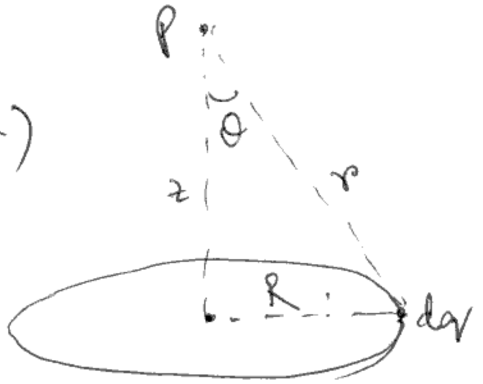
(b) $$\vec{E} = -\vec{\nabla}\phi = \phi_0 e^{-kz} k\,(\sin kx\,\hat{x} + \cos kx\,\hat{z})$$
(Show!)

(c) $$\int_S \vec{E}\cdot d\vec{a} = \frac{\sigma A}{\epsilon_0} \Rightarrow E_z = \frac{\sigma}{2\epsilon_0}.$$

$$\therefore \sigma = 2\epsilon_0 E_z = 2\epsilon_0 \phi_0 e^{-kz}\cos kx.$$

At $z=0$, $\sigma = 2\epsilon_0\phi_0\cos kx$.

4. (a)



At the pt P, magnitude of field due to charge dq is

$$\frac{dq}{4\pi\epsilon_0 r^2}.$$

Horizontal component of the field will cancel with the horizontal component of field due to charge dq at the diametrically opposite point. The vertical components add up.

$\therefore$ Total field at P is,

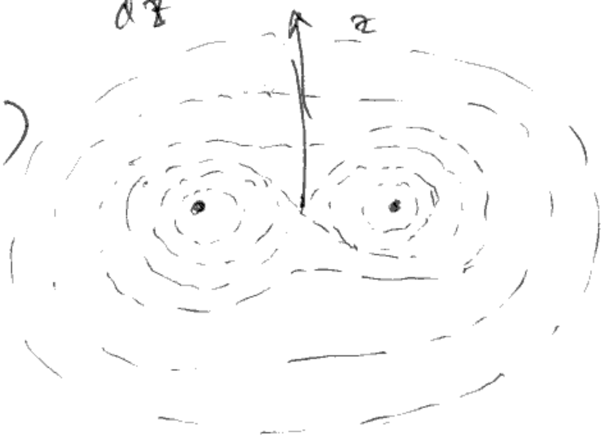
$$|\vec{E}| = \int \frac{dq\cos\theta}{4\pi\epsilon_0 r^2} = \int \frac{dq}{4\pi\epsilon_0 r^2}\left(\frac{z}{r}\right)$$

$$= \frac{Qz}{4\pi\epsilon_0 (z^2+R^2)^{3/2}} \quad (\because r = \sqrt{z^2+R^2}).$$

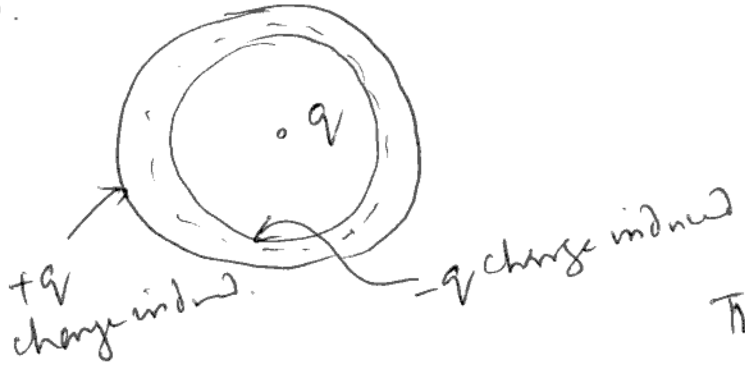
Limits: $z \to \infty$, $E(z) = Q/4\pi\epsilon_0 r^2$.
$z = 0$, $E(z) = 0$.

$\frac{dE_z}{dz} = 0 \Rightarrow$ Max^m at $z = \frac{R}{\sqrt{2}}$.

(b)



5.



Consider Gaussian surface that encloses charge q & the inner surface of the conducting shell. Then, if q_{ind} is the charge induced on the inner surface, then since $\vec{E} = 0$ inside the conductor,

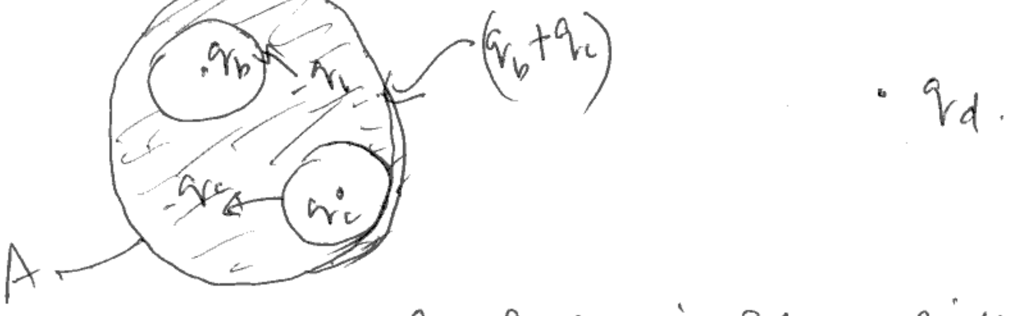
$$\oint \vec{E}\cdot d\vec{a} = 0 = \frac{q + q_{ind}}{\epsilon_0} \Rightarrow q_{ind} = -q.$$

$\therefore +q$ charge is induced on the outer surface.

We will assume q is distributed in a spherically symmetric manner (will be discussed later).

The field outside is due to the outer surface charge only. By Gauss' law, the field is radial with respect to the center of the shell and has magnitude $\frac{q}{4\pi\epsilon_0 r^2}$ which is the same as due to a charge q located at the center of the shell.

6.



Force on q_b & q_c is zero, since $\vec{E}=0$ inside conductor.

The induced charges follow from the argument in the previous problem.

The charge induced on the outer surface of A is (q_b+q_c) distributed in a spherically symmetric manner. The field due to A : $\dfrac{(q_b+q_c)}{4\pi\epsilon_0 r^2}$.

q_d will disturb the distribution of charge but not the amount ~~of~~ of charge on surface of A.

If q_d is placed far enough, then,

force on q_d = $q_d \cdot \left(\dfrac{q_b+q_c}{4\pi\epsilon_0 r^2}\right)$

$$\therefore \vec{F}_d = \frac{q_d(q_b+q_c)}{4\pi\epsilon_0 r^2}\hat{r}.$$

Force on A : $\vec{F}_A = -\vec{F}_d$.