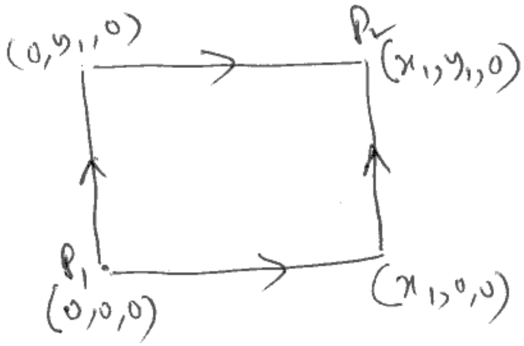
# Assignment 2.

1.




Path I :

$$\int_{P_1}^{P_2} \vec{E} \cdot d\vec{s} = \int_{(0,0,0)}^{(x_1,0,0)} \vec{E} \cdot d\vec{s} + \int_{(x_1,0,0)}^{(x_1,y_1,0)} \vec{E} \cdot d\vec{s}$$

$$\vec{E} \cdot d\vec{s} = 6xy\,dx + (3x^2 - 3y^2)\,dy.$$

$$\left.\begin{array}{l} y = 0 \text{ for } (0,0,0) \longrightarrow (x_1, 0, 0) \\ \& \ dy = 0 \end{array}\right\} \therefore \text{1st integral vanishes.}$$

$x = x_1$ & $dx = 0$ for $(x_1, 0, 0) \longrightarrow (x_1, y_1, 0)$.

$$\therefore \int_{(x_1,0,0)}^{(x_1,y_1,0)} \vec{E} \cdot d\vec{s} = \int_{(x_1,0,0)}^{(x_1,y_1,0)} (3x_1^2 - 3y^2)\,dy$$

$$= 3x_1^2 y_1 - y_1^3.$$

$$\therefore \int_{P_1}^{P_2} \vec{E} \cdot d\vec{s} = 3x_1^2 y_1 - y_1^3 \quad \text{for path I}$$

Show that for path II you get the same result

$$\therefore \phi = \int \vec{E} \cdot d\vec{s} + \text{const} = 3x^2 y - y^3 + \text{const}.$$

Now, $\vec{E} = -\vec{\nabla}\phi = \hat{x}\frac{\partial \phi}{\partial x} + \hat{y}\frac{\partial \phi}{\partial y} + \hat{z}\frac{\partial \phi}{\partial z}$ ~~(0 in this problem)~~

$$\therefore E_x = \frac{\partial \phi}{\partial x} = 6xy, \quad E_y = \frac{\partial \phi}{\partial y} = 3x^2 - 3y^2, \quad E_z = \frac{\partial \phi}{\partial z} = 0$$

~~2. The dimensions of $\phi$ seems a little messed up~~

2. Keeping in mind that we are using SI for our course, ~~we~~ it is better to include a factor of $\frac{\rho_0}{4\pi\epsilon_0}$ for the potential as,

$$\phi = \begin{cases} \frac{\rho_0}{4\pi\epsilon_0}(x^2+y^2+z^2) & x^2+y^2+z^2 < a^2 \\ \frac{\rho_0}{4\pi\epsilon_0}\left(-a^2 + \frac{2a^3}{(x^2+y^2+z^2)^{1/2}}\right) & x^2+y^2+z^2 > a^2 \end{cases}$$

$\rho_0$ has dimension of volume charge density.

Now, $\vec{E} = -\vec{\nabla}\phi = -\left(\hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z}\right)$

$$\frac{\partial\phi}{\partial x} = \frac{\rho_0}{4\pi\epsilon_0}\cdot 2x \quad \text{for } x^2+y^2+z^2 < a^2$$

$$= \frac{\rho_0}{4\pi\epsilon_0}\cdot 2a^3\left(-\frac{1}{\cancel{2}}\frac{\cancel{2}x}{(x^2+y^2+z^2)^{3/2}}\right) \quad \text{for } x^2+y^2+z^2 > a^2$$

$$= -\frac{\rho_0}{4\pi\epsilon_0}\cdot\frac{2a^3 x}{(x^2+y^2+z^2)^{3/2}}$$

$\therefore$ and similarly for $\frac{\partial\phi}{\partial y}$, $\frac{\partial\phi}{\partial z}$.

$$\therefore \vec{E} = -\vec{\nabla}\phi = -\frac{\rho_0}{4\pi\epsilon_0}(2x\hat{x} + 2y\hat{y} + 2z\hat{z})$$

$$= -\frac{2\rho_0}{4\pi\epsilon_0}(x\hat{x} + y\hat{y} + z\hat{z}) \quad \text{for } x^2+y^2+z^2 < a^2.$$

Similarly, $\vec{E} = +\frac{2\rho_0 a^3}{4\pi\epsilon_0}\cdot\frac{x\hat{x} + y\hat{y} + z\hat{z}}{(x^2+y^2+z^2)^{3/2}}$ for $x^2+y^2+z^2 > a^2$.

To find the charge distribution, we will calculate the Laplacian. $\nabla^2$.

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}.$$

For $x^2+y^2+z^2<a^2$, $\frac{\partial^2\phi}{\partial x^2} = \frac{\rho_0}{4\pi\epsilon_0}\cdot 2.$

Similarly for $\frac{\partial^2\phi}{\partial y^2}$, $\frac{\partial^2\phi}{\partial z^2}$.

$$\therefore \nabla^2\phi = \frac{\rho_0}{4\pi\epsilon_0}(2+2+2) = \frac{6\rho_0}{4\pi\epsilon_0} \text{ for } x^2+y^2+z^2<a^2$$

But, by Poisson's Eqn, $\nabla^2\phi = -\frac{\rho}{\epsilon_0}$.

where $\rho$ is the charge distribution.

$$\therefore \frac{\cancel{6}^{3}\rho_0}{\cancel{4}_{2}\pi\cancel{\epsilon_0}} = -\frac{\rho}{\cancel{\epsilon_0}} \Rightarrow \rho = -\frac{3\rho_0}{2\pi} \text{ for } x^2+y^2+z^2<a^2$$

For $x^2+y^2+z^2>a^2$, $\frac{\partial^2\phi}{\partial x^2} = -\frac{2\rho_0 a^3}{4\pi\epsilon_0}\cdot\frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right]$

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} = -\frac{2\rho_0 a^3}{4\pi\epsilon_0}\left[-\frac{3}{\cancel{2}}\cdot\frac{x\cdot \cancel{2}x}{(x^2+y^2+z^2)^{5/2}} + \frac{1}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= -\frac{2\rho_0 a^3}{4\pi\epsilon_0}\left[\frac{-3x^2}{(x^2+y^2+z^2)^{5/2}} + \frac{1}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= -\frac{2\rho_0 a^3}{4\pi\epsilon_0}\frac{1}{(x^2+y^2+z^2)^{3/2}}\left[\frac{-3x^2}{x^2+y^2+z^2} + 1\right].$$

Similarly for $\frac{\partial^2\phi}{\partial y^2}$, $\frac{\partial^2\phi}{\partial z^2}$.

$$\therefore \nabla^2\phi = -\frac{2\rho_0 a^3}{4\pi\epsilon_0(x^2+y^2+z^2)^{3/2}}\left[-\frac{3x^2}{x^2+y^2+z^2} + 1 - \frac{3y^2}{x^2+y^2+z^2} + 1 - \frac{3z^2}{x^2+y^2+z^2} + 1\right]$$

$$\therefore \nabla^2\phi = \frac{-2\rho_0 a^3}{4\pi\epsilon_0 (x^2+y^2+z^2)^{3/2}}\left[\frac{-3\cancel{(x^2+y^2+z^2)}}{\cancel{(x^2+y^2+z^2)}} + 3\right] = 0.$$

for $x^2+y^2+z^2 > a^2$.

$\therefore \rho = 0$ for $x^2+y^2+z^2 > a^2$.

• Note that just like probs. Assignment 1, there is a discontinuity in the electric field at the surface of the sphere since,

$$|\Delta\vec{E}| = |\vec{E}|\Big|_{\substack{x^2+y^2+z^2=a^2 \\ \text{outside}}} - |\vec{E}|\Big|_{\substack{x^2+y^2+z^2=a^2 \\ \text{inside}}} \neq 0.$$

→ implies existence of surface charge !!

This is best seen by going to spherical polar coordinates with $r^2 = x^2+y^2+z^2$.

$$\therefore \vec{E} = \frac{-2\rho_0}{4\pi\epsilon_0} r\hat{r} \quad \text{for } r < a.$$

$$= \frac{2\rho_0 a^3}{4\pi\epsilon_0}\cdot\frac{\hat{r}}{r^2} \quad \text{for } r > a.$$

$$|\Delta\vec{E}| = \frac{\rho_0}{4\pi\epsilon_0}\left[2a - (-2a)\right] = \frac{4a\rho_0}{4\pi\epsilon_0} = \frac{\rho_0 a}{\pi\epsilon_0}.$$

Since, $|\Delta\vec{E}| = \frac{\sigma}{\epsilon_0}$ with $\sigma$: surface charge density.

$$\therefore \frac{\sigma}{\epsilon_0} = \frac{\rho_0 a}{\pi\epsilon_0} \Rightarrow \sigma = \frac{\rho_0 a}{\pi}.$$

3. Let $R$ be the radius of the basketball.

Then, $\dfrac{Q}{4\pi\epsilon_0 R} = -1000 \Rightarrow Q = -4\pi\epsilon_0 R \cdot 1000.$

$$\text{Charge}/m^2 = \frac{Q}{4\pi R^2} = -\frac{\cancel{4\pi}\epsilon_0 R \cdot 1000}{\cancel{4\pi} R^2}$$

$$= -\frac{1000\,\epsilon_0}{R}.$$

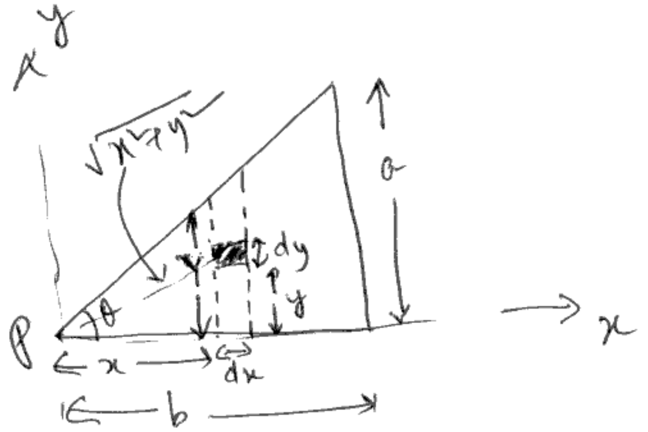
$\therefore$ # of extra electrons/$m^2$ = $\dfrac{\text{Charge}/m^2}{\text{electronic charge}}$

$$= \frac{1000\,\epsilon_0/R}{e}.$$

Assuming $R \simeq 0.15m$.

$$\# = \frac{8.85\times10^{-12}\times 1000}{1.6\times10^{-19}\times 0.15} \simeq 3.7\times10^{11}/m^2.$$

4.




To do this, we need to first divide the triangle into strips as shown & find the contribution of a strip at pt. P.

The potential at point P due to an infinitesimal area $dxdy$ (shaded) is,

$$d\phi' = \frac{\sigma\, dx\, dy}{4\pi\epsilon_0\sqrt{x^2+y^2}}.$$

$\therefore$ Potential due to the strip,

$$d\phi = \int_{y=0}^{y=Y} d\phi' = \frac{1}{4\pi\epsilon_0}\int_{y=0}^{y=Y} \frac{\sigma\, dx\, dy}{\sqrt{x^2+y^2}}$$

Now, $\frac{Y}{x} = \frac{a}{b}$. (Similar triangles). $\Rightarrow Y = \frac{ax}{b}$.

$$\therefore d\phi = \frac{1}{4\pi\epsilon_0}\int_{y=0}^{y=ax/b} \frac{\sigma\, dx\, dy}{\sqrt{x^2+y^2}} = \frac{\sigma\, dx}{4\pi\epsilon_0}\int_0^{ax/b} \frac{dy}{\sqrt{x^2+y^2}}.$$

$$= \frac{\sigma\, dx}{4\pi\epsilon_0}.\left[\ln\left(y+\sqrt{x^2+y^2}\right)\right]_0^{ax/b}$$ (You can do the integration)

$$= \frac{\sigma\, dx}{4\pi\epsilon_0}.\left[\ln\left(\frac{ax}{b}+\sqrt{x^2+\frac{a^2x^2}{b^2}}\right) - \ln x\right]$$

$$= \frac{\sigma\, dx}{4\pi\epsilon_0}\ln\left(\frac{a}{b}+\sqrt{1+\frac{a^2}{b^2}}\right)$$

$\tan\theta = \frac{a}{b}$ $\quad\therefore\quad d\phi = \frac{\sigma\, dx}{4\pi\epsilon_0}\ln\left(\tan\theta + \sec\theta\right).$

$$= \frac{\sigma\, dx}{4\pi\epsilon_0}\ln\left(\frac{1+\sin\theta}{\cos\theta}\right).$$

$\therefore$ Contribution at P due to whole triangle,

$$\phi = \int_{x=0}^{x=b} d\phi = \frac{\sigma}{4\pi\epsilon_0}\ln\left(\frac{1+\sin\theta}{\cos\theta}\right)\int_0^b dx.$$

$$= \frac{\sigma b}{4\pi\epsilon_0}\ln\left(\frac{1+\sin\theta}{\cos\theta}\right).$$

5. Total energy = Energy of shell 1 + Energy of shell 2 + Energy of one shell due to potential of the other.

Energy of shell 1 = Energy of shell 2 = $\frac{Q^2}{8\pi\epsilon_0 R}$.

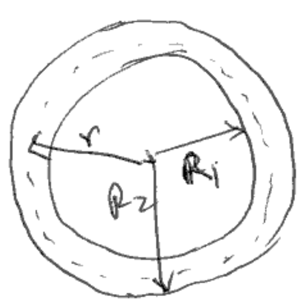
Energy required to build one shell given that the other shell is present $= Q\phi$

where $\phi$ = potential due to a shell = $\dfrac{Q}{4\pi\epsilon_0 R}$

$\therefore$ Total energy = $\dfrac{Q^2}{8\pi\epsilon_0 R} + \dfrac{Q^2}{8\pi\epsilon_0 R} + Q\phi$.

$$= \frac{Q^2}{4\pi\epsilon_0 R} + Q\cdot\left(\frac{Q}{4\pi\epsilon_0 R}\right) = \frac{Q^2}{2\pi\epsilon_0 R}$$

6. (a) Already shown (Assignment 1),




for $0 \le r \le R_1$, $E(r) = 0$ (electric field inside shell)

This is a thick shell not a thin one

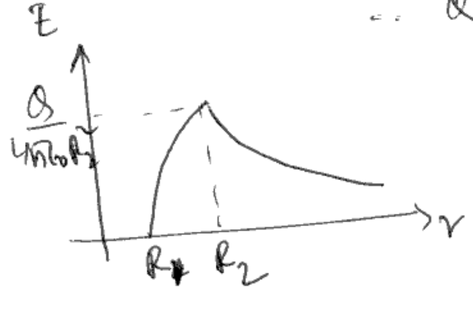
The electric field inside the shell;

$$E = \frac{Q_r}{4\pi\epsilon_0 r^2} \quad \text{for } R_1 \le r \le R_2.$$

To find $Q_r$ note that $Q = \rho\cdot\frac{4}{3}\pi\left(R_2^3 - R_1^3\right)$

where $\rho$ = charge density.

$$\therefore Q_r = \rho\cdot\frac{4}{3}\pi\left(r^3 - R_1^3\right) = \frac{Q\,\cancel{\frac{4}{3}\pi}\left(r^3-R_1^3\right)}{\cancel{\frac{4}{3}\pi}\left(R_2^3-R_1^3\right)}$$




$$= Q\cdot\frac{r^3 - R_1^3}{R_2^3 - R_1^3}$$

$$\therefore E = \frac{Q}{4\pi\epsilon_0}\cdot\frac{1}{R_2^3 - R_1^3}\left(\frac{r^3 - R_1^3}{r^2}\right) \quad R_1 \le r \le R_2.$$

For $R_2 \le r \le \infty$, $E(r) = \dfrac{Q}{4\pi\epsilon_0 r^2}$.

(b) Potential at $r=0$ is,

$$\phi(0) = -\int_\infty^0 E\,dr = -\left[\int_\infty^{R_2} + \int_{R_2}^{R_1} + \int_{R_1}^{0}\right] E\,dr$$

$$\therefore \phi(0) = -\int_{\infty}^{R_2} \frac{Q}{4\pi\varepsilon_0 r^2}\,dr - \int_{R_1}^{R_2} \frac{Q}{4\pi\varepsilon_0 (R_2^3 - R_1^3)} \cdot \frac{r^3 - R_1^3}{r^2}\,dr$$

$$= \frac{9}{14} \frac{Q}{4\pi\varepsilon_0 R}$$ (choosing $R_2 = 2R_1 = 2R$) i.e. $R = R_1$.

Please do the integration!