Vectors Solutions

A.1) $\vec{a} = 3\hat{i} + 12\hat{j} + 4\hat{k}$
$\therefore$ Unit vector in the direction of $\vec{a}$

$$= \frac{\vec{a}}{\mid \vec{a} \mid} = \frac{3\hat{i} + 12\hat{j} + 4\hat{k}}{\sqrt{3^2 + 12^2 + 4^2}} = \frac{1}{13}(3\hat{i} + 12\hat{j} + 4\hat{k})$$

A.2) $\vec{a} = 3\hat{i} + 4\hat{j}$
$\vec{b} = 4\hat{i} + 4\hat{j} + 2\hat{k}$
$\vec{a} \cdot \vec{b} = \mid \vec{a} \mid \ \mid \vec{b} \mid \ cos\theta \therefore \vec{a} \cdot \vec{b} = (3\hat{i} + 4\hat{j}) \cdot (4\hat{i} + 4\hat{j} + 2\hat{k}) = 12 + 16 = 28$
$\mid \vec{a} \mid = \sqrt{3^2 + 4^2} = 5$
$\mid \vec{b} \mid = \sqrt{4^2 + 4^2 + 2^2} = 6$

$$\therefore 28 = 30 \ cos\theta$$
$$\Longrightarrow cos\theta = \frac{14}{15}$$
$$\Longrightarrow \theta = cos^{-1}\left(\frac{14}{15}\right)$$

A.3) Projection of $\vec{A}$ on $\vec{B} = \frac{(\vec{A} \cdot \vec{B})}{|\vec{B}|}$
$\vec{A} = 3\hat{i} + 2\hat{j}$
$\vec{B} = 4\hat{i} - \hat{j}$

$$Projection = \frac{(3\hat{i} + 2\hat{j}) \cdot (4\hat{i} - \hat{j})}{\sqrt{4^2 + 1^2}}$$
$$= \frac{12 - 2}{\sqrt{17}} = \frac{10}{\sqrt{17}}$$

A.4) $\mid \vec{A} \mid = 3, \mid \vec{B} \mid = 8$
$\mid \vec{A} \times \vec{B} \mid = 12$

$$\therefore \ \mid \vec{A} \times \vec{B} \mid = \mid \vec{A} \mid \mid \vec{B} \mid \ sin\theta$$
$$\Longrightarrow 12 = 3 \ sin\theta$$
$$\Longrightarrow sin\theta = \frac{1}{2}$$

A.5) $\vec{A} = 3\hat{i} + 4\hat{j} - 4\hat{k}$. Take any vector $\vec{B}$ in the x-y plane.Then, $\vec{B}$ is given by

$$\vec{B} = a\hat{i} + b\hat{j}$$

Since $\vec{B}$ is perpendicular to $\vec{A}$,

$$\vec{A}.\vec{B} = 0$$
$$3a + 4b = 0$$
$$a = -\frac{4b}{3}$$

We can take any value of a and b which satisfies this equation. So, choose b=3 and a=-4. Then, $\vec{B} = -4\hat{i} + 3\hat{j}$ is one such vector.

A.6) If $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$, show that $\vec{a}\perp\vec{b}$.

$|\vec{a}+\vec{b}|=\sqrt{|\vec{a}|^2+|\vec{b}|^2+2|\vec{a}||\vec{b}|\ cos\theta}$

$|\vec{a}-\vec{b}|=\sqrt{|\vec{a}|^2+|\vec{b}|^2-2|\vec{a}||\vec{b}|\ cos\theta}$

$$\therefore \sqrt{|\vec{a}|^2+|\vec{b}|^2+2|\vec{a}||\vec{b}|\ cos\theta}=\sqrt{|\vec{a}|^2+|\vec{b}|^2-2|\vec{a}||\vec{b}|\ cos\theta}$$
$$\implies |\vec{a}|^2+|\vec{b}|^2+2|\vec{a}||\vec{b}|\ cos\theta=|\vec{a}|^2+|\vec{b}|^2-2|\vec{a}||\vec{b}|\ cos\theta$$
$$\implies 4|\vec{a}||\vec{b}|\ cos\theta=0$$
$$\implies cos\theta=0$$
$$\therefore \theta=\frac{\pi}{2}$$

$\therefore \vec{a}\perp\vec{b}$

A.7) i) Prove

$$(\vec{a}\times\vec{b})\cdot\vec{c}=\begin{vmatrix}c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$LHS=(\vec{a}\times\vec{b})\cdot\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\cdot(c_1\hat{i}+c_2\hat{j}+c_3\hat{k})$$
$$=\begin{vmatrix}\hat{i}(c_1\hat{i}+c_2\hat{j}+c_3\hat{k}) & \hat{j}(c_1\hat{i}+c_2\hat{j}+c_3\hat{k}) & \hat{k}(c_1\hat{i}+c_2\hat{j}+c_3\hat{k})\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$
$$=\begin{vmatrix}c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=RHS$$

ii) $(2\hat{i}+3\hat{j})\cdot[(2\hat{i}+3\hat{j})\times(4\hat{j}+7\hat{k})]$

This is of form

$$=[\vec{a}\ \vec{b}\ \vec{c}]$$
$$=\begin{vmatrix}2 & 3 & 0\\ 2 & 3 & 0\\ 0 & 4 & 7\end{vmatrix}$$
$$=0$$

iii) Prove that $[\vec{a}\ \vec{b}\ \vec{c}]$ is cyclic, i.e $[\vec{a}\ \vec{b}\ \vec{c}]=[\vec{b}\ \vec{c}\ \vec{a}]=[\vec{c}\ \vec{a}\ \vec{b}]$.

$$(\vec{a}\times\vec{b})\cdot\vec{c}=\begin{vmatrix}c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$(\vec{b} \times \vec{c}) \cdot \vec{a} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(\vec{c} \times \vec{a}) \cdot \vec{b} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

Upon calculating the determinant they all turn out equal.

A.8) i) Prove $[\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$.

$$\begin{aligned} LHS &= [\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}] \\ &= (\vec{a}+\vec{b}) \cdot \{(\vec{b}+\vec{c}) \times (\vec{c}+\vec{a})\} \\ &= (\vec{a}+\vec{b}) \cdot \{\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a}\} \\ &= (\vec{a}+\vec{b}) \cdot \{\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a}\} \\ &= (\vec{a}+\vec{b}) \cdot (\vec{b} \times \vec{c}) + (\vec{a}+\vec{b}) \cdot (\vec{b} \times \vec{a}) + (\vec{a}+\vec{b}) \cdot (\vec{c} \times \vec{a}) \\ &= \vec{a} \cdot (\vec{b} \times \vec{c}) + 0 + \vec{b} \cdot (\vec{c} \times \vec{a}) \\ &= [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{b}\ \vec{c}\ \vec{a}] \\ &= [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{b}\ \vec{c}] \qquad \left\{ \because [\vec{a}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{b}\ \vec{c}] \right\} \\ &= 2[\vec{a}\ \vec{b}\ \vec{c}] = RHS \end{aligned}$$

ii) Prove $[\vec{a} \times \vec{b} \quad \vec{b} \times \vec{c} \quad \vec{c} \times \vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$.

$$\begin{aligned} LHS &= (\vec{a} \times \vec{b}) \cdot \{(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a})\} \\ &= (\vec{a} \times \vec{b}) \cdot \{[\vec{b}\ \vec{c}\ \vec{a}]\vec{c} - [\vec{b}\ \vec{c}\ \vec{c}]\vec{a}\} \end{aligned}$$

$[\vec{b}\ \vec{c}\ \vec{c}] = 0$ because $[\vec{b}\ \vec{c}\ \vec{c}]$ has two rows equal and determinant of matrix having two rows equal is zero.

$$\begin{aligned} LHS &= (\vec{a} \times \vec{b}) \cdot [\vec{b}\ \vec{c}\ \vec{a}]\vec{c} \\ &= \{(\vec{a} \times \vec{b}) \cdot \vec{c}\}[\vec{b}\ \vec{c}\ \vec{a}] \\ &= [\vec{a}\ \vec{b}\ \vec{c}][\vec{b}\ \vec{c}\ \vec{a}] \\ &= [\vec{a}\ \vec{b}\ \vec{c}]^2 = RHS \end{aligned}$$