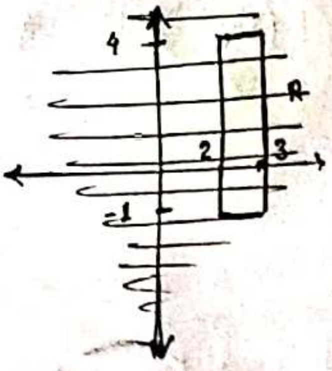
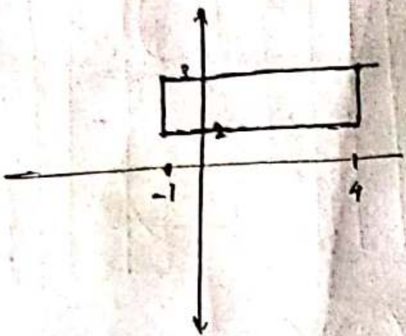
DOUBLE INTEGRALS



① $\iint_R (12x-18y).dA$.

ⓐ $\int_2^3 \left(\int_{-1}^4 (12x-18y).dx\right).dy$

when x varies from -1 to 4
y is treated as constant



$= \int_2^3 [6x^2-18xy]_{-1}^4 .dy$

$= \int_2^3 6[16-1]-18y[4+1].dy$

$= \int_2^3 90-90y.dy$ [Now y varies from 2 to 3, so y is variable here]

$= [90y-45y^2]_2^3$

$= 90(3-2)-45(9-4)$

$= 90-225$

$= -135$.

ⓑ Just interchange dx and dy in the prev. and proceed similarly.

② $\iint_R (2x-4y^3).dA$ R: $-5 \le x \le 4$, $0 \le y \le 3$.

$\int_0^3 \left(\int_{-5}^4 (2x-4y^3).dx\right).dy$

$= \int_0^3 [x^2-4xy^3]_{-5}^4 .dy$

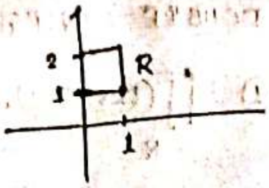
$= \int_0^3 [(16-25)-4(4+5)y^3].dy$

$= [-9y-9y^4]_0^3$

$-9(3-0)-9(81-0)$

$= -9\times 84 = -756$.

3. $$\iint_R \frac{1}{(2x+3y)^2}\,.dA \qquad R: 0 \le x \le 1,\; 1 \le y \le 2.$$



$$= \int_1^2 \left(\int_0^1 \frac{dx}{(2x+3y)^2} \right).dy$$

$$= \int_1^2 \left[\frac{-1}{2[2x+3y]} \right]_0^1 .dy$$

$$= \int_1^2 \left[\frac{-1}{2(2+3y)} + \frac{1}{6y} \right].dy$$

$$= \frac{1}{6} \int_1^2 \frac{1}{y} - \frac{1}{y+\frac{2}{3}}\,.dy$$

$$= \frac{1}{6} \left[\log\left(\frac{y}{y+\frac{2}{3}} \right) \right]_1^2$$

$$= \frac{1}{6} \left[\ln\left(\frac{2}{8/3} \right) - \ln\left(\frac{1}{5/3} \right) \right]$$

$$= \frac{1}{6} \left[\ln(3/4) - \ln(3/5) \right]$$

$$= \frac{1}{6} \ln(5/4)$$

4. $$\iint_R x.\cos^2 y.\,dA \qquad R: -2 \le x \le 3,\; 0 \le y \le \frac{\pi}{2}.$$

$$= \int_0^{\pi/2} \left(\int_{-2}^3 x.\cos^2 y.\,dx \right).dy$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{x^2}{2} \cos^2 y \right]_{-2}^3 .dy.$$

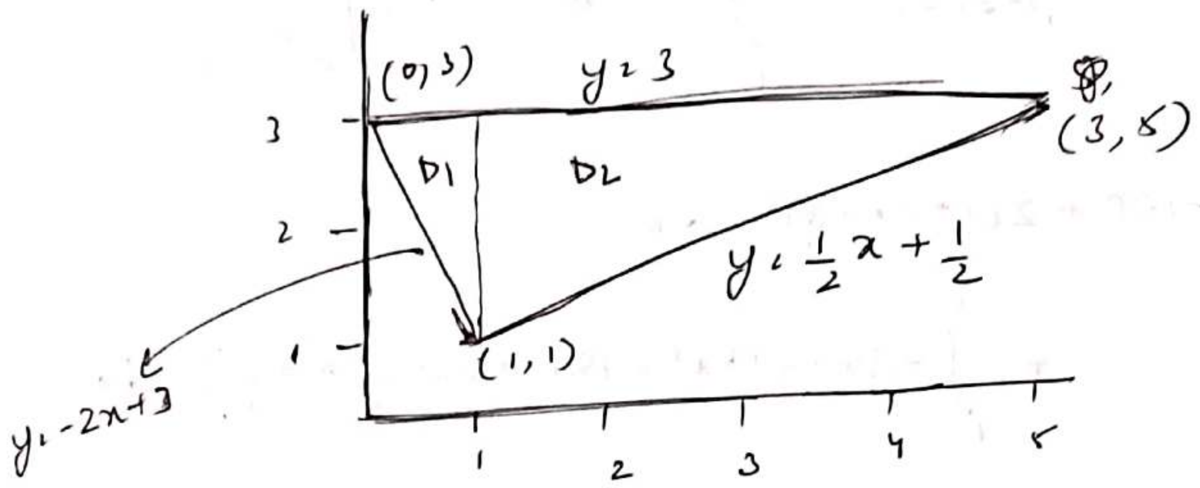
$$= \int_0^{\pi/2} \frac{9-4}{2}.\frac{1}{2}.(1+\cos 2y).dy.$$

$$= \frac{5}{4} \left[y + \frac{\sin 2y}{2} \right]_0^{\pi/2}$$

$$= \frac{5}{4} \left[\frac{\pi}{2} - 0 + \frac{1}{2}(\sin\pi - \sin 0) \right]$$

$$= \frac{5\pi}{8}.$$

5) $\iint_D (6x^2-40y)\,dA$, D is the triangle with vertices (0,3), (1,1), (5,3)



$D_1 = \{(x,y) \mid 0 \le x \le 1,\ -2x+3 \le y \le 3\}$

$D_2 = \{(x,y) \mid 1 \le x \le 5,\ \frac{1}{2}x + \frac{1}{2} \le y \le 3\}$

$\boxed{D = D_1 \cup D_2}$

$$\iint_D (6x^2-40y)\,dA = \iint_{D_1} 6x^2-40y\,dA + \iint_{D_2} 6x^2-40y\,dA$$

The entire triangle has an area D, for ease of integration, I have divided D into D_1 & D_2. Now, we have found limits separately for each region, so we can write it in the usual form now

$$\iint_D (6x^2-40y)\,dA =$$

$$= \int_0^1 \int_{-2x+3}^{3} (6x^2-40y)\,dy\,dx + \int_1^5 \int_{\frac{1}{2}x+\frac{1}{2}}^{3} (6x^2-40y)\,dy\,dx.$$

$$= \int_0^1 \left(6x^2y - 20y^2\right)\Big|_{-2x+3}^{3} dx + \int_1^5 \left(6x^2y - 20y^2\right)\Big|_{\frac{1}{2}x+\frac{1}{2}}^{3} dx$$

$$= \int_0^1 \left(12x^3 - 180 + 20(-2x+3)^2\right) dx$$

$$+ \int_1^5 -3x^3 + 18x^2 - 180 + 20\left(\frac{1}{2}x + \frac{1}{2}\right)^2 dx$$

$$= \left[3x^4 - 180x - \frac{10}{3}(3-2x)^3\right]_0^1 + \left(-\frac{3}{4}x^4 + 5x^3 - 180x + \frac{40}{3}\left(\frac{1}{2}x + \frac{1}{2}\right)^3\right)\Big|_1^5$$

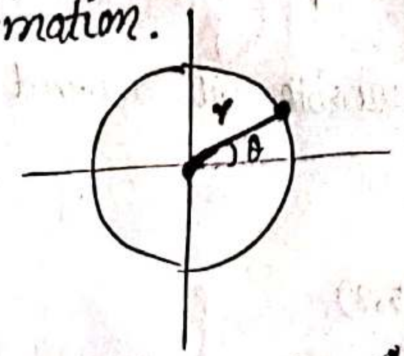
$$= -\frac{935}{3} = \underline{\underline{-311.67}}$$

6. $\iint_D e^{x^2+y^2} \cdot dA = S$ D – unit disk centred at origin.

Here we need to perform a polar transformation.

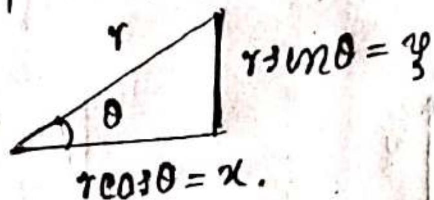
$x = r\cos\theta$

$y = r\sin\theta$



$dA = r.dr.d\theta$

[Remember this for mean time]



$\therefore x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta)$

$= r^2$.

$$S = \int_0^{2\pi}\left(\int_0^1 e^{r^2}.r.dr\right).d\theta$$

substitute $u = r^2$ $r=1 \to u=1$, $r=0 \to u=0$

$du = 2r.dr$.

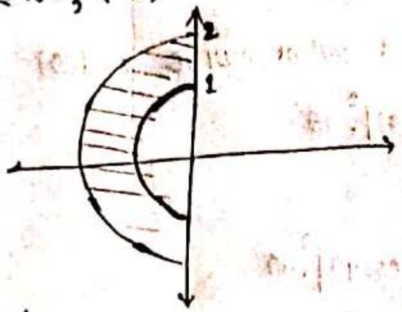
$$\int_0^{2\pi}\left(\int_0^1 e^u \cdot \frac{du}{2}\right).d\theta$$

$$= \int_0^{2\pi}\left[\frac{e^u}{2}\right]_0^1 . d\theta$$

$$= \left(\frac{e}{2} - 1\right) \times 2\pi$$

$$= \pi(e-2).$$

7. $\iint_R (x+y).\,dA$ $\qquad R: 1 \le x^2+y^2 \le 4,\ x \le 0$



use polar transformation

$x = r\cos\theta$

$y = r\sin\theta$

$1 \le r^2 \le 4$

$\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ [measured anticlockwise wrt x axis]

$\cancel{\int_1^2 r(\cos\theta)}.$ $\quad \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left[\int_1^2 r(\cos\theta + \sin\theta).\,r.\,dr.\right] d\theta$

$$= \int_{\frac{\pi}{2}}^{3\pi/2} \left[\frac{r^3}{3}(\cos\theta + \sin\theta)\right]_1^2 .\,d\theta.$$

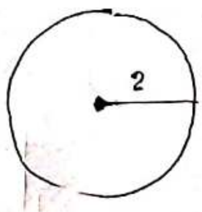
$$= \int_{\frac{\pi}{2}}^{3\pi/2} \left(\frac{8-1}{3}\right)(\cos\theta + \sin\theta).\,d\theta$$

$$= \frac{7}{3}\left[\sin\theta - \cos\theta\right]_{\frac{\pi}{2}}^{3\pi/2}$$

$$= \frac{7}{3}(-1-0-1+0)$$

$$= \frac{-14}{3}.$$

8. S. $\iint_R (4-x^2-y^2).\,dA$ $\qquad$ R - circle of radius 2.



$x = r\cos\theta$

$y = r\sin\theta.$

$$S = \int_0^{2\pi}\left(\int_0^2 (4-r^2)\,r.\,dr\right).\,d\theta$$

$$= \int_0^{2\pi}\left[\int_0^2 4r - r^3.\,dr\right].\,d\theta$$

$$= \int_0^{2\pi}\left[2r^2 - \frac{r^4}{4}\right]_0^2 .\,d\theta$$

since r and θ are indepndent variable we can integrate them seperately

$$= \left[2[4-0] - \frac{16-0}{4}\right].\int_0^{2\pi} d\theta$$

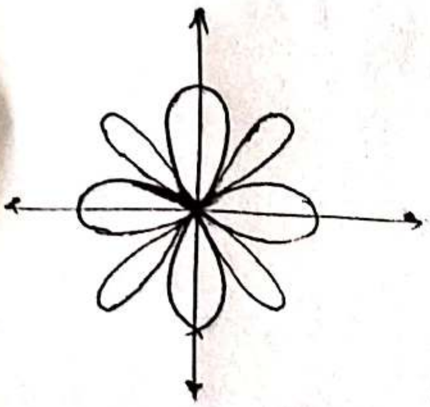
$$= (8-4)\times 2\pi$$

$$= 8\pi$$

9. Area of the curve $r = \cos 4\theta$.

Formula for area A of a polar curve $r = f(\theta)$

is $$A = \int_a^b \frac{1}{2}[f(\theta)]^2 \, d\theta$$



$r = \cos 4\theta$.

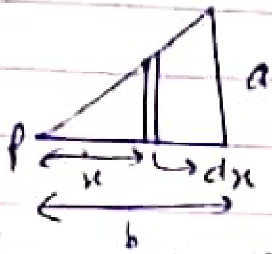
$$= \int_0^{2\pi} \frac{1}{2}[\cos 4\theta]^2 \, d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}(1 + \cos 8\theta)\, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{\sin 8\theta}{8}\right]_0^{2\pi}$$

$$= \frac{\pi}{2}.$$

10)



$$dV = \frac{dq}{4\pi\varepsilon_0 r} \Rightarrow V = \iint \frac{\sigma\, dx\, dy}{4\pi\varepsilon_0 r}$$

$$\tan\theta = \frac{a}{b} = \frac{y}{x} \Rightarrow y = \frac{ax}{b}$$

$$V = \int_0^b \int_0^{\frac{ax}{b}} \frac{\sigma\, dx\, dy}{4\pi\varepsilon_0 \sqrt{x^2+y^2}}$$

$$= \int_0^b \underbrace{\frac{\sigma}{4\pi\varepsilon_0}\left(\int_0^{\frac{ax}{b}} \frac{dy}{\sqrt{x^2+y^2}}\right)}_{\text{Contribution of } dx} dx$$

$$= \int_0^b \frac{\sigma}{4\pi\varepsilon_0}\left[\log\left|y+\sqrt{x^2+y^2}\right|\right]_0^{\frac{ax}{b}} dx$$

$$= \frac{\sigma}{4\pi\varepsilon_0}\int_0^b \log\left|\frac{\frac{ax}{b}+\sqrt{x^2+\frac{a^2x^2}{b^2}}}{0+\sqrt{x^2+0}}\right| dx$$

$$= \frac{\sigma}{4\pi\varepsilon_0}\int_0^b \log\left|\frac{a}{b}+\sqrt{1+\frac{a^2}{b^2}}\right| dx$$

$$= \frac{\sigma}{4\pi\varepsilon_0}\log\left|\frac{a}{b}+\sqrt{1+\frac{a^2}{b^2}}\right| \int_0^b dx$$

$$= \frac{\sigma b}{4\pi\varepsilon_0}\log\left|\frac{a}{b}+\sqrt{1+\frac{a^2}{b^2}}\right|$$