Complex Numbers - Solutions

Maths Workshop

1 Problems 1.1

Question 1.1 : $-2-i$, $3+2i$
Sum $= (-2-i)+(3+2i) = (-2+3)+i(-1+2) = 1+i$
Product $= (-2-i)(3+2i) = -6+i(-4-3)-2i^2 = (-6+2)+i(-4-3) = -4-i$
Question 1.2 : $1-i^3$, $\dfrac{1}{(1-i)^3}$
$1-i^3 = 1-i(i^2) = 1+i$

$$\frac{1}{(1-i)^3} = \frac{(1+i)^3}{(1-i)^3(1+i)^3} = \frac{1+3i+3i^2+i^3}{[(1-i)(1+i)]^3} = \frac{(1+3i-3-i}{2^3} = \frac{(-2+2i}{8} = \frac{-1+i}{4}$$

Sum $= (1+i)+\left(\dfrac{-1+i}{4}\right) = \dfrac{(4-1)+i(4+1)}{4} = \dfrac{3}{4}+i\dfrac{5}{4}$

Product $= (1+i)\left(\dfrac{-1+i}{4}\right) = \dfrac{(1+i)(-1+i)}{4} = \dfrac{-1+i^2}{4} = \dfrac{-1-1}{4} = -\dfrac{2}{4} = -\dfrac{1}{2}$

Question 1.3 : $1+i$, $1-i$
Sum $= (1+i)+(1-i) = (1+1)+i(1-1) = 2$
Product $= (1+i)(1-i) = 1-i^2 = 1-(-1) = 1+1 = 2$

Question 2.1 : $2+5i$
Conjugate $= 2-5i$
Inverse $= \dfrac{1}{2+5i} = \dfrac{2-5i}{(2+5i)(2-5i)} = \dfrac{2-5i}{4-25i^2} = \dfrac{2-5i}{29} = \dfrac{2}{29}+i\left(-\dfrac{5}{29}\right)$

Question 2.2 : $3+2i^3$
$3+2i^3 = 3+2i(i^2) = 3-2i$
Conjugate $= 3+2i$
Inverse $= \dfrac{1}{3-2i} = \dfrac{3+2i}{(3-2i)(3+2i)} = \dfrac{3+2i}{9-4i^2} = \dfrac{3+2i}{13} = \dfrac{3}{13}+i\left(\dfrac{2}{13}\right)$

Question 2.3 : i
$i = 0+i$
Conjugate $= 0-i = -i$
Inverse $= \dfrac{1}{i} = \dfrac{-i}{i(-i)} = \dfrac{-i}{-i^2} = -i$

Question 3 : Give an example of a complex number that does not have a multiplicative inverse.
Answer : $0 = 0+0i$. Can we have other complex numbers ?

2 Problems 2.1

Question 1.1 : Which symbol represents $|z|$?
Answer : r

Question 1.2 : Which symbol represents $\text{amp}(z)$?
Answer : θ

Question 1.3 : Find a relation between x, y and r.

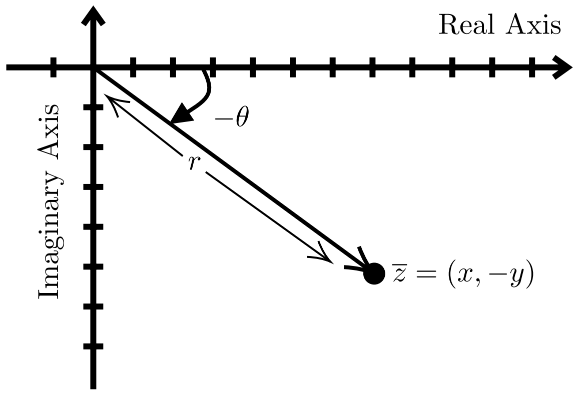
Answer : By Pythagoras theorem, we have $x^2 + y^2 = r^2$.

Question 1.4 : Find a relation between x, r and θ.
Answer : From the right-angled triangle, we have $\frac{x}{r} = \cos\theta \implies x = r\cos\theta$.

Question 1.5 : Find a relation between y, r and θ.
Answer : From the right-angled triangle, we have $\frac{y}{r} = \sin\theta \implies y = r\sin\theta$.

Question 1.6 : Find a relation between x, y and θ.
Answer : From the last two relations, we have $\frac{y}{x} = \tan\theta$

Question 1.7 : Plot $\overline{z}$.
Answer :



Question 1.8 : Verify that the complex number can be written as $z = r(\cos\theta + i\sin\theta$.
Answer : We have $x = r\cos\theta$ and $y = r\sin\theta$. So, $z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$.

Question 2.1 : $2 + 5i$
Using the relations that we have deduced above, we have
Modulus $= \sqrt{2^2 + 5^2} = \sqrt{29}$
Amplitude $= \theta$ such that $\tan\theta = \frac{5}{2}$

Question 2.2 : $\frac{1}{2+5i}$

$\frac{1}{2+5i} = \frac{2-5i}{(2+5i)(2-5i)} = \frac{2}{29} + i\left(-\frac{5}{29}\right)$

Modulus $= \sqrt{\left(\frac{2}{29}\right)^2 + \left(-\frac{5}{29}\right)^2} = \sqrt{\frac{4+25}{29^2}} = \sqrt{\frac{1}{29}}$

Amplitude $= \theta$ such that $\tan\theta = \frac{-5/29}{2/29} = -\frac{5}{2}$

Question 2.3 : $2 - 5i$
Modulus $= \sqrt{2^2 + (-5)^2} = \sqrt{29}$
Amplitude $= \theta$ such that $\tan\theta = -\frac{5}{2}$

Question 3 : Find a relation between z, $\overline{z}$ and $|z|$.
Answer : Let $z = x + iy$. Then $\overline{z} = x - iy$ and $|z| = \sqrt{x^2 + y^2}$. $z\overline{z} = (x + iy)(x - iy) = x^2 - i^2y^2 = x^2 + y^2 = |z|^2$.
So, $z\overline{z} = |z|^2$.

Question 4 : Find a relation between z^{-1}, $\overline{z}$ and $|z|$.
Answer : Let $z = x+iy$. Then $\overline{z} = x-iy$ and $|z| = \sqrt{x^2 + y^2}$. $z^{-1} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2} = \frac{\overline{z}}{|z|^2}$.
So, $z^{-1} = \frac{\overline{z}}{|z|^2}$.

Question 5 : Find a relation between $|z|$, $|\overline{z}|$ and $|z^{-1}|$.

Answer : Let $z = x+iy$. Then $\overline{z} = x-iy$ and $z^{-1} = \dfrac{\overline{z}}{|z|^2}$. So, $|z| = \sqrt{x^2+y^2}$ and $|\overline{z}| = \sqrt{x^2+(-y)^2} = \sqrt{x^2+y^2}$. So, $|z| = |\overline{z}|$. Now, $|z^{-1}| = \dfrac{|\overline{z}|}{|z|^2} = \dfrac{|z|}{|z|^2} = \dfrac{1}{|z|}$. So, $|z| = |\overline{z}| = \dfrac{1}{|z^{-1}|}$.

3 Problems 3.1

Question 1.1 : $e^{i\pi}$
Answer : $e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0i = -1$

Question 1.2 : $e^{i\frac{\pi}{2}} \cdot e^{i\frac{\pi}{4}}$
Answer : $e^{i\frac{\pi}{2}} \cdot e^{i\frac{\pi}{4}} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = i\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}i}\right) = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}i^2 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

Question 1.3 : $e^{i\frac{\pi}{3}} + e^{i\frac{\pi}{4}}$
Answer :

$$
\begin{aligned}
e^{i\frac{\pi}{3}} + e^{i\frac{\pi}{4}} = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) + \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) &= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \\
&= \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right) + i\left(\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\right) \\
&= \frac{1+\sqrt{2}}{2} + \frac{\sqrt{2}+\sqrt{3}}{2}i
\end{aligned}
$$

Question 2.1 : $\left(\dfrac{3}{2} + \dfrac{3\sqrt{3}}{2}i\right)^3$
Answer :

$$
\begin{aligned}
\left(\frac{3}{2} + \frac{3\sqrt{3}}{2}i\right)^3 &= 3^3\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3 \\
&= 3^3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3 \\
&= 3^3\left(\cos\pi + i\sin\pi\right) \\
&= -27
\end{aligned}
$$

Question 2.2 : $\left(\dfrac{1+i}{\sqrt{2}}\right)^4$
Answer :

$$
\begin{aligned}
\left(\frac{1+i}{\sqrt{2}}\right)^4 &= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^4 \\
&= \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^4 \\
&= (\cos\pi + i\sin\pi) \\
&= -1
\end{aligned}
$$

Question 2.3 : $(1+i)^{10}$
Answer :

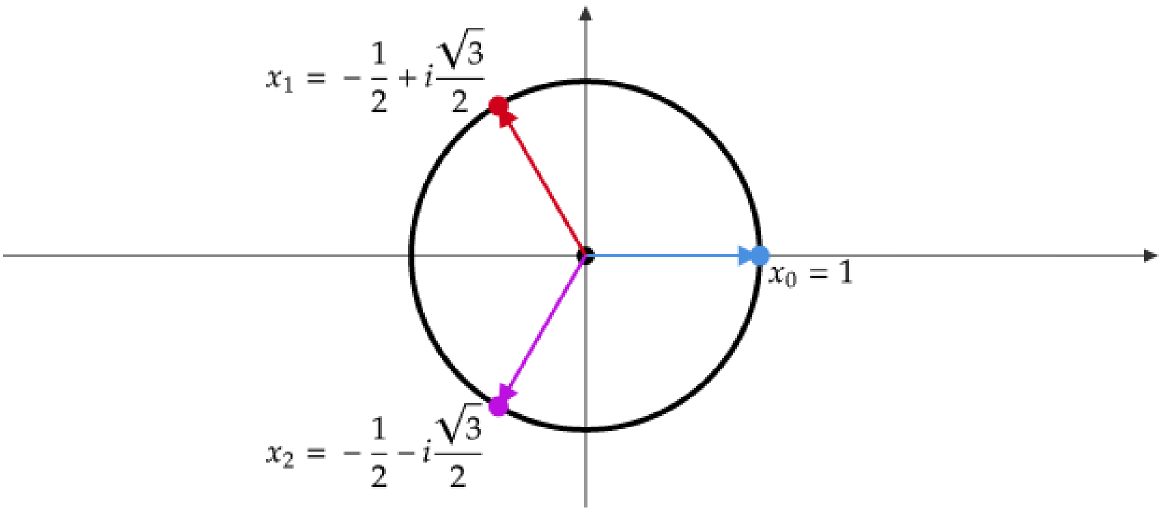
$$
\begin{aligned}
(1+i)^{10} &= (\sqrt{2})^{10})\left(\frac{1+i}{\sqrt{2}}\right)^{10} \\
&= 32\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{10} \\
&= 32\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right) \\
&= 32\left(\cos\left(2\pi + \frac{\pi}{2}\right) + i\sin\left(2\pi + \frac{\pi}{2}\right)\right) \\
&= 32\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \\
&= 32i
\end{aligned}
$$

Question 3 : Calculate the roots of $x^3 - 1 = 0$ and plot them.
Answer : We will follow the process described previously.

- Step 1 : $x^3 - 1 = 0 \implies x^3 = 1$.
- Step 2 : $z = e^{i\theta}$ is a root of the above equation, where θ is a real number.
- Step 3 : Using de Moivre's theorem and properties of trigonometric funcions, we get $\theta_k = \dfrac{2k\pi}{3}$ and corresponding solutions $x_k = e^{i\theta_k}$ for $k = 0, 1, 2$.
- Step 4 : We list the solutions explicitly :

$$x_0 = e^{i0} = \cos 0 + i \sin 0 = 1$$
$$x_1 = e^{i\frac{2\pi}{3}} = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$
$$x_2 = e^{i\frac{4\pi}{3}} = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$

We plot the solutions as follows :



Question 4 : Verify that the complex roots are squares of each other.
Answer : One way to verify would be to compute the squares and check. We will try to verify it in a different way, using de Moivre's theorem and Euler's formula.
Check that $\left(e^{i\frac{2\pi}{3}}\right)^2 = e^{i\frac{4\pi}{3}}$.
Now, $\left(e^{i\frac{4\pi}{3}}\right)^2 = e^{i\frac{8\pi}{3}} = e^{i\left(2\pi + \frac{2\pi}{3}\right)} = \cos\left(2\pi + \frac{2\pi}{3}\right) + i \sin\left(2\pi + \frac{2\pi}{3}\right) = \cos \dfrac{2\pi}{3} + i \sin \dfrac{2\pi}{3} = e^{i\frac{2\pi}{3}}$.
So, we see that the complex roots are squares of each other.

Question 5 : Denote the roots by $1, \omega$ and ω^2. Calculate $1 + \omega + \omega^2$.
Answer : Direct calculations show $1 + \omega + \omega^2 = 1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = 0$. Alternatively, we know that ω is a root of the equation $x^3 - 1 = 0$. So, $\omega^3 - 1 = 0$. Factorising the expression on L.H.S. yields $(\omega - 1)(1 + \omega + \omega^2) = 0$. We know that $\omega \neq 1$, hence, we must have $1 + \omega + \omega^2 = 0$