BEFORE YOU START WRITING, check the page numbers at the bottom to confirm that all the pages are present. You have **ONE** hour to complete this exam. **You must explain your work clearly to get credit for your answer.** Use the available space judiciously.

Name: ABHIGYAN W MEDHI Reg. No: MS17108 Tutorial section: T3

Question:	1	2	3	4	5	Total
Points:	6	3	3	2	6	20
Score:	6	2.5	3	2	6	19.5

1. People arrive at a queue according to the following scheme: During each minute of time either 0 or 1 person arrives. In a minute, the probability that 1 person arrives is p and that no person arrives is $q = 1 - p$. Answer the following questions (No explanation required):

(2 marks) (a) Let C be the number of customers arriving in the first 10 minutes.

- $P(C = 2) =$ $\binom{10}{2} p^2 (1-p)^8$
- $E[C] =$ $10p$

2

(2 marks) (b) Let W be the time (in minutes) until the first person arrives.

- $P(W = 5) =$ $(1-p)^4 p^1$
- $E[W] =$ $\frac{1}{p}$

2

(2 marks) (c) Let T be the time (in minutes) until 4 people arrive.

- $P(T = 10) =$ $\binom{9}{3} p^4 (1-p)^6$
- $E[T] =$ $\frac{4}{p}$

2

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(3 marks) 2. Consider two independent random variables: $X \sim Poi(\lambda)$ and $Y \sim Poi(\mu)$ for $\lambda, \mu > 0$. Determine the probability mass function of $Z = X + Y$.

2/ $X \sim Poi(\lambda)$ $\quad Y \sim Poi(\mu)$ $\quad \lambda, \mu > 0$

let $Z = X + Y$

$$P(Z=k) = P\left\{ \bigcup_{r+s=k} \{X=r\} \cap \{Y=s\} \right\}$$

$$= P\left\{ \bigcup_{r=0} \{X=r\} \cap \{Y=k-r\} \right\}$$

$$= \sum_{r=0}^{k} P(\{X=r\} \cap \{Y=k-r\})$$

$$= \sum_{r=0}^{k} P(X=r)\, P(Y=k-r)$$

$$= \sum_{r=0}^{k} e^{-\lambda} \frac{\lambda^r}{r!} \cdot e^{-\mu} \frac{\mu^{k-r}}{(k-r)!}$$

2.5

$> 0.$

3. Let $X \sim Unif([0,1])$.

marks) (a) Determine the probability density function of the random variable X^2.

1 mark) (b) Compute $E[X^2]$.

3/ $X \sim unif \ [0,1]$ $f_X(x) \begin{cases} 0 & x < 0 \\ 1 & 0 < x < 1 \\ 0 & x > 1 \end{cases}$

$F_{X^2}(x) = P\{X^2 < x\}$

$= P\{-\sqrt{x} < X < \sqrt{x}\}$

$= \int_{-\sqrt{x}}^{\sqrt{x}} f_X(x)\,dx$

$= 1\int_{0}^{\sqrt{x}} dx \quad$ For $0 < x < 1$

2

$= [x]_0^{\sqrt{x}} = \sqrt{x}$

$\frac{d F_{X^2}(x)}{dx} = f_{X^2}(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}}$ for $0 < x < 1$

$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$

$= \int_0^1 x^2\,dx$

$= \left[\frac{x^3}{3}\right]_0^1$

1

$= 1/3$

(2 marks) 4. Let $Y \sim \mathcal{N}(3,9)$ and $\phi(x) = P(Z \leq x)$, where $Z \sim \mathcal{N}(0,1)$. Compute $P(Y > 3|Y > 1)$ in terms of $\phi(2/3)$.

$Y \sim N(3,9) \qquad \phi(x) = P(z \leq x)$

$$P(Y>3 | Y>1) = \frac{P(\{Y>3\} \cap \{Y>1\})}{P(\{Y>1\})}$$

$$= \frac{P(Y>1 | Y>3)\, P(Y>3)}{P(Y>1)}$$

$$= \frac{P(Y>3)}{P(Y>1)}$$

$Y > 3$

$\Rightarrow \frac{Y-3}{3} > \frac{3-3}{3}$

$\Rightarrow z > 0$

$$= \frac{P(z>0)}{P(z>-2/3)}$$

$Y > 1$

$\Rightarrow \frac{Y-3}{3} > -\frac{2}{3}$

$\Rightarrow z > -2/3$

$$= \frac{1 - P(z<0)}{1 - P(z<-2/3)}$$

$$= \frac{1/2}{1-(1-P(z<2/3))}$$

$$= \frac{1/2}{\phi(2/3)}$$

$$= \frac{1}{2\phi(2/3)}$$

2

5. Define $C(X,Y) = E[(X - E[X])(Y - E[Y]]$. Show that:

(2 marks) (a) $C(X,Y) = E[XY] - E[X]E[Y]$.

(2 marks) (b) For all $a, b \in \mathbb{R}$, $a^2E[X^2] + 2abE[XY] + b^2E[Y^2] \geq 0$.

(2 marks) (c) $E[XY]^2 \leq E[X^2]E[Y^2]$.

(Hint: For $A, B, C \in \mathbb{R}$, $Ar^2 + 2Br + C \geq 0\ \forall r \in \mathbb{R}$ implies $B^2 \leq AC$)

(a) $C(X,Y) = E[(X - E[X])(Y - E[Y])]$

we know $E[aX] = aE(X)$ if a = const

$= E[E(XY) - E[X]Y - XE[Y] + E[X]E[Y]]$

$= E[XY] - E[X]E[Y] - \cancel{E[X]E[Y]} + \cancel{E[X]E[Y]}$

$= E[XY] - E[X]E[Y]$ //

2

(b) $(aX + bY)^2 \geq 0 \quad \forall \ a, b \in R$

$\Rightarrow E[(aX + bY)^2] \geq 0$

$\Rightarrow E[a^2X^2 + b^2Y^2 + 2ab\,XY] \geq 0$

$\Rightarrow a^2E[X^2] + 2ab\,E[XY] + b^2E[Y^2] \geq 0$ // — ①

2

(c) Dividing ① by b^2, we get

$\left(\frac{a}{b}\right)^2 E[X^2] + 2\left(\frac{a}{b}\right)E[XY] + E[Y^2] \geq 0$ $\quad r = \left(\frac{a}{b}\right) \in R$ if $b \neq 0$

$\Rightarrow r^2E[X^2] + 2rE[XY] + E[Y^2] \geq 0$

$\therefore (E[XY])^2 \leq E[X^2]E[Y^2]$ [From Hint]

2

$\leftarrow Ar^2 + 2Br + C \geq 0$

$\sqrt{(2B)^2 - 4AC} \leq 0$

$\Rightarrow B^2 - AC \leq 0$

$\Rightarrow B^2 \leq AC$

$\Rightarrow$

(2)

PROBABILITY AND STATISTICS (MTH-202)

16/01/2019

Name: ABHIGYAN W MEDHI (T3)

Registration number: MS17108

Time: 15 minutes | QUIZ-I | Maximum Marks: 4

1. Three distinct coins are flipped simultaneously. Let H_i (for $i = 1, 2, 3$) denote the event that i^{th} coin shows Heads.

   (a) Describe the sample space. What is the size of event H_1?

   (b) Write the event $E = \{\text{All coins land Tails}\}$ in terms of H_i's.

[2]

a) $\Omega = \{(HHH), (HHT), (THH), (HTH), (HTT), (TTH), (THT), (TTT)\}$

$H_1 = \{(HHH), (HHT), (HTH), (HTT)\}$

(1) $|H_1| = 4$

b) $E = \{\text{All coins land Tails}\}$ in terms H_i's

$E = \cancel{H_1^c \cap H_2^c \cup H_3^c}\ H_1^c \cap H_2^c \cap H_3^c$

(1)

or

$E = (H_1 \cup H_2 \cup H_3)^c$

2. Let A, B be events in a sample space Ω. Prove that if $P(A) = P(B) = 0$, then $P(A \cup B) = 0$. [2]

Ans Let $E_1 = \Omega$ $E_2 = \phi$

And $E_1 \cap E_2 = \phi$

$P(E_1 \cup E_2) = P(E_1) + P(E_2)$ [By 3rd Axiom]

$\Rightarrow$ $P(\Omega) = P(\Omega) + P(\phi)$

$\Rightarrow$ $P(\phi) = 0$

Now, given $P(A) = P(B) = 0$

$\Rightarrow A = B = \phi$

0

$A \cap B = \phi$

$\Rightarrow P(A \cap B) = P(\phi)$

$= 0$

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0$

PROBABILITY AND STATISTICS (MTH-202)

23/01/2019

Name: ABHIGYAN W. MEDHI (T3)

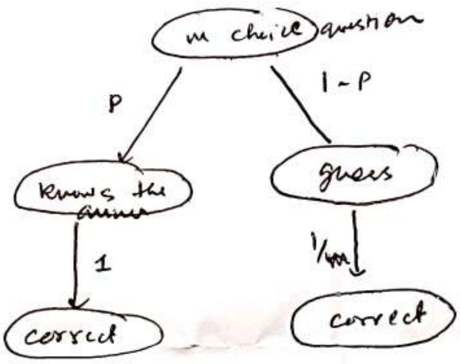
Registration number: MS17108

Time: 15 minutes | QUIZ-2 | Maximum Marks: 4

1. Consider a multiple choice question with m choices. The student either knows the answer with probability p or guesses it with probability $1-p$. Probability that the guess is correct of $1/m$. What is the probability that the student knew the answer given that it has been answered correctly? [2]

Ans P (~~a~~

$P(A/B) = \frac{P(A\cap}{P(B)}$



~~P(correct)~~ ~~of~~ E = correct answer given

F = knows the answer

$P(E) = p.1 + (1-p)\cdot\frac{1}{m}$

$\cancel{P(E/F) = \frac{P(E\cap F)}{P(F)}} = \quad P(F/E) = \frac{P(F\cap E)}{P(E)}$

$= \frac{p}{p + (1-p)\, 1/m}$

2

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2. Pick a number randomly from $\{1, 2, \ldots, 10\}$ (all outcomes are equally likely). Let E denote the event that the number picked is an even number and F denote the event that it is an odd number. Compute the following probabilities:

(i) $P(E) =$ 5/10 = 1/2

(ii) $P(F) =$ 1/2

(iii) $P(E \cap F) =$ 0

(iv) $P(E \cup F) =$ 1

[2]

2

PROBABILITY AND STATISTICS (MTH-202)
30/01/2019

4

Name: ABHIGYAN W. MEDHI
Registration number: MS17108

Time: 15 minutes QUIZ-4 Maximum Marks: 4

1. Suppose X is a discrete uniform random variable taking values in $\{a, a+1, a+2, \ldots, b\}$ for some $0 < a \le b$. Compute the Expectation of X.
(Hint: Assume $b = a + k$, for some $k \ge 0$)

[2]

1. $S = \{a, a+1, \ldots, b\}$

$|S| = b - a + 1$

$X \sim \text{uniform}(|S|)$

$P(a) = P(a+1) = \ldots = \frac{1}{b-a+1} = \frac{1}{k+1}$

$b = a + k$

$\therefore E[X] = \frac{1}{k+1}\{a + (a+1) + (a+2) + \ldots (a+k)\}$

$= \frac{1}{k+1}\left\{\frac{k+1}{2}\{2a + k d\}\right\}$

$= \frac{1}{2}(2a + k d)$

$= \frac{1}{2}(2a + k)$

2

$3 + 3+1 + 3+2 + 3+3 \; 3+4$

k_1

$1 \; 2 \; 3 \; 4 \; 5$

$\frac{6}{2}[2 + 3d]$

5.6

$\frac{n+1}{2}\{2a + (n-1)d\}$

$(k+1)a + 1 + 2 \ldots k$

$\frac{k(k+1)}{2}$

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2. Suppose an urn contains 5 white and 3 black balls. Balls are drawn from the urn at random (independently each time), the colour of the ball drawn is noted and the ball is replaced in the urn.

(a) What is the probability of getting a white ball for the first time on the 3rd draw?

$P(W) = (1-p)^2 p = \left(\frac{3}{8}\right)^2 \frac{5}{8}$

1

(b) What is the probability of drawing a white ball on 3rd draw?

$P(W \text{ on 3rd draw}) = 5/8$

1

[2]

Tutorial Section: T3

PROBABILITY AND STATISTICS (MTH-202)

30/01/2019

4

Name: Abhigyan W. Hedli

Registration number: MS17108

Time: 15 minutes | QUIZ-5 | Maximum Marks: 4

1. Let X and Y be two continuous random variables with joint density function:

$$f_{XY}(x,y) = \begin{cases} Ce^{-(x+3y)} & \text{for } x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Answer the following questions:

1. Find C.
2. Compute $P(X > 1, Y < 5)$

[1+1=2]

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy$$

2

$$1 = \int_0^{\infty}\int_0^{\infty} c\, e^{-x} e^{-3y}\, dx\, dy$$

$$1 = -c\left[e^{-x}\right]_0^{\infty} \int_0^{\infty} e^{-3y}\, dy\, dx$$

$$1 = -\frac{c}{3}\left[e^{-3y}\right]_0^{\infty}$$

$$\Rightarrow \quad 3 = c$$

$$P(X>1, Y<5) = 3\int_1^{\infty}\int_0^{5} e^{-x} e^{-3y}\, dx\, dy$$

$$= +\frac{3}{3}\left[e^{-x}\right]_1^{\infty}\left[e^{-3y}\right]_0^{5}$$

$$= \left[0 - e^{-1}\right]\left[e^{-15} - e^{0}\right]$$

$$= e^{-1}\left(1 - e^{-15}\right)$$

2. Suppose that a lecture hall has two doors. Let X be the number of people who entered through door 1 and Y be the number of people who entered through door 2 in one hour. Assume that $X \sim Poi(\lambda)$, $Y \sim Poi(\mu)$ and that X and Y are independent. Let $N = X + Y$ denote the total number of people who entered the hall in one hour. Compute:

1. $E[N]$.
2. $Var(N)$.

[1+1=2]

2/ $X \sim Poi(\lambda)$ $Y \sim Poi(\mu)$

$N = X + Y$

$E[N] = E[X] + E[Y]$ ∵ They are independent

$= \lambda + \mu$

$Var(N) = Var(X) + Var(Y)$

$= \lambda + \mu$

Poison r.v has same variance and mean

$f_Y(y) = e^{-\mu} \frac{\mu^y}{y!}$

$f_X(x) = e^{-\lambda} \frac{\lambda^x}{x!}$

$E[X] = e^{-\lambda} \sum_{k=0}^{\infty} \frac{k \lambda^k}{k!}$

$E[X] = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$

$E[X] = \lambda e^{-\lambda} e^{\lambda}$

$E[X] = \lambda$

2

~~$E[(x+y)^2]$~~

~~$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)^2 f_x(x) f_y(y)\, dx\, dy$~~

~~$x^2 f_x(x) + y^2 f_y(y) + \int 2xy f_x f_y$~~

$Var(X+Y) = E[(X+Y)^2] - (E[X] + E[Y])^2$

$= E[X^2] + E[Y]^2 + 2E[XY]$

$- E[X]^2 - E[Y]^2$

$- 2E[X]E[Y]$

$= Var(X) + Var(Y)$ //

k^2

$(k \cdot 1 + 1)$

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Tutorial Section: T3

PROBABILITY AND STATISTICS (MTH-202)
30/01/2019

Name: SANAT M.
Registration number: MS17107

4

Time: 15 minutes QUIZ-5 Maximum Marks: 4

1. Let X and Y be two continuous random variables with joint density function:

$$f_{XY}(x,y) = \begin{cases} Ce^{-(x+3y)} & \text{for } x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Answer the following questions:

1. Find C.
2. Compute $P(X > 1, Y < 5)$

2 [1+1=2]

1) $\int_0^\infty \int_0^\infty Ce^{-(x+3y)}\,dx\,dy = 1$

$\Rightarrow C\int_0^\infty e^{-3y}\,dy \int_0^\infty e^{-x}\,dx = 1$

$\Rightarrow C\int_0^\infty e^{-3y}\,dy = 1 \quad \Rightarrow -\frac{C}{3}(-1) = 1$

$\therefore \underline{C = 3}$

2) $3\int_1^\infty \int_0^5 e^{-(x+3y)}\,dy\,dx = 3\int_1^\infty e^{-x}\,dx \cdot \left(-\frac{1}{3}\right) e^{-3y}\Big|_0^5$

$= e^{-x}\Big|_1^\infty \cdot (e^{-15} - 1)$

$= \underline{(1 - e^{-15})} \cdot e^{-1}$

2. Suppose that a lecture hall has two doors. Let X be the number of people who entered through door 1 and Y be the number of people who entered through door 2 in one hour. Assume that $X \sim Poi(\lambda)$, $Y \sim Poi(\mu)$ and that X and Y are independent. Let $N = X + Y$ denote the total number of people who entered the hall in one hour. Compute:

1. $E[N]$.
2. $Var(N)$.

[1+1=2]

1) $E[N]$

$= E[X+Y] \cancel{= E[X] + E[Y]}$ ~~since X and~~ Y ~~are independent~~

$\cancel{E[X] = \sum_{i=1}^{\infty} \frac{e^{-\lambda} \cdot \lambda^i}{i!}}$

$E[X+Y] \Rightarrow$ first calculating $X+Y$

$$X+Y = P\{X=k, Y=N-k\}$$

$$= \sum_{k=1}^{n} P\{X=k\} \cdot P\{Y=N-k\}$$

$$= \sum_{k=1}^{n} \frac{e^{-\lambda}\lambda^k}{k!} \cdot \frac{e^{-\mu}\mu^{n-k}}{(n-k)!}$$

$$= e^{-(\lambda+\mu)} \sum_{k=1}^{n} \frac{\lambda^k \mu^{n-k}}{k!(n-k)!} \cdot \frac{n!}{n!}$$

$$= \frac{e^{-(\lambda+\mu)}}{n!} \sum_{k=1}^{n} \frac{\lambda^k \mu^{n-k} \cdot n!}{k!(n-k)!} = \frac{e^{-(\lambda+\mu)}}{n!}(\lambda+\mu)^n$$

(Poisson itself)

2

$E[X+Y] = \cancel{\left(\frac{1}{\lambda+\mu}\right)} (\lambda+\mu)$

2) $\cancel{Var(N) = E[N^2] - (E[N])^2}$

$\cancel{= E[X^2+Y^2+2XY] - \left(\frac{1}{\lambda+\mu}\right)^2 (\lambda+\mu)^{2}}$

$=$

$Var(N) = \lambda+\mu$ (mean = variance for Poisson distribution)