MTH202: Assignment 8

February 25, 2019

- **Convolution:** Let X and Y be two discrete random variables and let $Z = X + Y$. Then, the probability mass function of Z is given by the convolution of the mass functions of X and Y, given by:

$$P(Z = k) = \sum_{r=0}^{k} P(X = r)P(Y = k - r)$$

- **Expectation of a continuous random variable:** Let X be a real-valued continuous random variable with probability density f_X. Then, the expectation of X is given by:

$$E[X] = \int_{-\infty}^{\infty} f_X(x)dx$$

Exercises

1. An insurance company has 1000 policies on people of age 60. The company estimates that the probability that a person of age 60 dies within a year is 0.01. What is the number of claims that the company can expect from beneficiaries of these men within a year.

2. Check that convolution of probability mass functions is commutative and associative.

3. A die is rolled twice (independently each time) with outcomes X, Y. Let $Z = \min(X, Y)$. Find the cumulative distribution function of Z in terms of cumulative distribution functions of X and Y.

4. A fair die is rolled until the first time T that a six turns up. Compute the following:

   (a) $P(T > 3)$.

   (b) $P(T > 6|T > 3)$.

5. If a coin is tossed a independently several times, what is the probability that the first head will occur after the fifth toss, given that it has not occurred in the first two tosses?

6. Consider two discrete random variables X and $Z \sim Poi(\lambda)$ such that $P(X = i|Z = k) = \binom{k}{i} p^i (1-p)^{k-i}$. What is $P(X = 0)$?

7. Let X be a real-valued random variable and $F_X(a) = P(X \leq a)$ denote its cumulative distribution function. Then, prove the following:

   (a) F_X is a non-decreasing function.

   (b) $\lim_{a \to \infty} F_X(a) = 1$.
   (Hint: consider a sequence of real numbers $b_n \to \infty$ and events $\{X \leq b_n\}$).

   (c) $\lim_{a \to -\infty} F_X(a) = 0$.

   (d) F_X is right continuous.
   (Hint: consider a sequence of real numbers $b_n \to b$ and events $\{X \leq b_n\}$).

   (e) $P(s < X \leq t) = F_X(t) - F_X(s)$.
   (Hint:Write the event $\{X \leq t\}$ as a disjoint union of two other events.)

8. For what values of constant C do these functions define a probability density on $\mathbb{R}$.

   (a)
   $$f_X(x) = \begin{cases} C(x^3 - x) & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (b) $f_X(x) = Ce^{-x^2/2} \quad \text{for} \quad x \in \mathbb{R}$.

   (c)
   $$f_X(x) = \begin{cases} Ce^{-x/10} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

9. Consider the following probability density functions and compute the Expectation.

   (a)
   $$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

   (b)
   $$f_X(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(c)

$$f_X(x) = \begin{cases} 3x^2 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(d)

$$f_X(x) = \begin{cases} 1/x & \text{for } 1 \leq x \leq e \\ 0 & \text{otherwise} \end{cases}$$

10. Consider a clock and let X denotes the time starting from $t = 0$ when the alarm goes off. Suppose $P(X \leq t) = \frac{e^{-t/4}}{4}$. What is the expected waiting time until the alarm goes off.