MTH202: Assignment 4

January 26, 2019

Independent events: Two events E, F are called independent if $P(E \cap F) = P(E)P(F)$.

Discrete (real-valued) Random Variable: Random variables are real-valued functions defined on the sample space. A random variable that can take at most countably many possible values is called discrete.

p.mf. of a discrete random variable X: Suppose X takes values in the set $\{x_1, x_2, \ldots\}$. Then, the probability mass function of X is defined as: $p_i = p(x_i) = P(X = x_i)$. We have:

$$p(x_i) \geq 0 \text{ for } i = 1, 2, \ldots \quad \text{and} \quad \sum_{i=1}^{\infty} p(x_i) = 1$$

Expectation and Variance of a discrete Random Variable:

$$E[X] = \sum_{i=1}^{\infty} x_i p(x_i).$$

$$Var(X) = E[(X - E[X])^2].$$

Exercises

1. Suppose A and B are independent events. Show that A^c and B^c are also independent.

2. A, B, C be independent events. Show that A and $B \cap C$ are independent.

3. A fair coin is tossed three times. Consider the following events:

   - $E = \{$Toss 1 and toss 2 produce different outcomes$\}$
   - $F = \{$Toss 2 and toss 3 produce different outcomes$\}$
   - $G = \{$Toss 3 and toss 1 produce different outcomes$\}$

   Show that $P(E) = P(E|F) = P(E|G)$ but $P(E) \neq P(E|F \cap G)$.

4. Suppose two candidates A and B are running against each other in an election, and A wins (or is tied) by receiving $m \geq n$ votes, where n is the number of votes received by B. What is the probability that A stays ahead of B throughout the voting if $m = n$? For $m > n$, suppose $P_{m,n}$ denotes the probability that A stays ahead of B throughout the voting. Write a recursion for $P_{m,n}$.

5. Two fair dice are rolled. Let X be the larger of the two numbers shown. Compute $P(\{X \in [2, 4]\})$.

6. Let X be the number of tosses of a fair coin until it shows Heads. Find $P(\{X \in 2\mathbb{N}\})$, where $2\mathbb{N}$ denotes the set of even numbers.

7. A die is rolled twice. What is the expectation of the sum of outcomes.

8. Show that for a random variable $X \geq 0$, then $E[X] \geq 0$.

9. Show that for a random variable $X \geq 0$, $E[X] = 0$ implies $P(\{X = 0\}) = 1$.

10. X be the number shown when a single fair die is rolled. Compute $E[X]$.

11. Let X be a random variable taking values in $\mathbb{N}$. Consider the following probabilities assigned to each value i: $p(i) = P(X = i) = \frac{1}{2^i}$. Show that this is a probability mass function. Compute the expectation.

12. Let X be a discrete random variable with p.m.f.

$$P(k) = \begin{cases} 0.2 & \text{for } k = 0 \\ 0.2 & \text{for } k = 1 \\ 0.3 & \text{for } k = 2 \\ 0.3 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

Define $Y = X(X-1)(X-2)$. Find the p.m.f. of Y.

13. Let X be a discrete random variable with p.m.f.

$$P(k) = \begin{cases} 0.1 & \text{for } k = 0 \\ 0.4 & \text{for } k = 1 \\ 0.3 & \text{for } k = 2 \\ 0.2 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

Compute the following:

- Find $E[X]$.
- Find $Var(X)$.
- If $Y = (X-2)^2$, find $E[Y]$.