MTH202: Assignment 11

April 5, 2019

- Recall that $E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$.
- Suppose $X_1, X_2, \ldots, X_n$ are independent random variables then $Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i)$.
- **Moment generating function of a random variable** X

$$M_X(t) = E\left[e^{tX}\right]$$

  Then, the n^{th} derivative of M_X evaluated at 0, $M_X^{(n)}(0) = E[X^n]$.

- **Markov's Inequality:** Let X be a non-negative random variable with finite expectation. Then, for any $a > 0$

$$P(X \geq a) \leq \frac{E[X]}{a}$$

- **Chebyshev's Inequality:** Late X be a random variable with finite expectation μ and variance σ^2, then for any value of $b > 0$,

$$P(|X - \mu| \geq b) \leq \frac{\sigma^2}{b^2}$$

- **Weak Law of Large Numbers:** Let $X_1, X_2, \ldots, X_n$ be i.i.d random variables with finite expectation $E[X_i] = \mu$. Then, for any $\epsilon > 0$,

$$\lim_{n\to\infty} P\left(\left|\frac{X_1 + X_2 + \ldots + X_n}{n} - \mu\right| \geq \epsilon\right) = 0$$

Exercises

1. Consider a function $h : (a, b) \to \mathbb{R}$ such that for any $x_1, \ldots, x_n \in (a, b)$ and for any $p_1, \ldots, p_n \geq 0$ such that $\sum_{i=1}^{n} p_i = 1$, we have

$$h\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i h(x_i) \tag{1}$$

Consider a random variable X that takes n different values in (a, b). Show that:

$$h(E[X]) \leq E[h(X)] \tag{2}$$

2. Verify that $g(x) = x^2$ satisfied (1). Give an alternative proof to show that (2) is satisfied by $g(x) = x^2$.

3. Let X, Y be independent random variables such that $E[X] = E[Y] = 2$, $Var(X) = -1$ and $Var(Y) = 3$. Compute the following:
   - $E[X + Y]$
   - $E[X^2], E[Y^2]$
   - $Var(X + Y)$
   - $E[(X + Y)^2]$

4. A fair coin is tossed repeatedly. Suppose that HEADS appears for the first time after X tosses and TAILS appears first time after Y tosses. Find the joint probability mass function of X and Y. Compute the corresponding marginals.

5. Show that $X + Y \sim Poi(\lambda + \mu)$, where $X \sim Poi(\lambda)$ and $Y \sim Poi(\mu)$ are independent random variables, by computing the moment generating function of X and Y and using $M_{X+Y}(t) = M_X(t)M_Y(t)$.

6. Let $X \sim Exp(\lambda)$. Compute $M_X(t)$ for $t < \lambda$. Compute $E[X^n] = M_X^{(n)}(0)$, where $M_X^{(n)}$ denotes the n^{th} derivative with respect to t.

7. Let $X \sim Exp(\lambda), Y \sim Exp(\mu)$ be independent random variables. Compute the probability density function of:
   - $Z = X + Y$
   - $W = \min(X, Y)$

8. Let $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ for $1 = 1, 2, \ldots, n$ be independent random variables. Compute the expectation and variance of $\sum_{i=1}^{n} X_i$. What is the probability density function of $\sum_{i=1}^{n} X_i$?

9. Let $Y = \sum_{i=1}^{N} X_i$, where X_i, N are independent random variables and X_i are identically distributed. Show that $E[Y] = E[N]E[X_1]$.
   (Hint: Proceed by computing the moment generating function of Y)

10. Consider an unfair coin with probability p of getting HEADS. Let S_n be the number of HEADS obtained when the coin is tossed repeatedly and independently n times. Show that, for any $\epsilon > 0$

$$\lim_{n \to \infty} P\left(\left|\frac{S_n}{n} - p\right| > \epsilon\right) = 0$$

11. Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. random variables with expectation μ and variance v. Define $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then, show that:

- Compute $E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2\right]$
- Show that $\lim_{n\to\infty} P\left(\left|\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 - v\right| > \epsilon\right) = 0$ for any $\epsilon > 0$.

12. A fair coin is tossed independently n times. Let S_n be the number of HEADS obtained. Use Chebyshev's inequality to find a lower bound of the probability that S_n/n differs from $1/2$ by less than 0.1 when $n = 100$ and $10,000$ and $100,000$.

13. Let X be a random variable such that $E[X] = 0$ and $P(-3 < X < 2) = 1/2$. Find a lower bound for $Var(X)$.

14. Let $X \sim Exp(\lambda)$. Using Markov's inequality find an upper bound for $P(X \geq a)$ for some $a > 0$. Compare the upper bound with the actual value of $P(X \geq a)$.

15. Let X_i be i.i.d. $Unif(0, 1)$. We define the sample mean as

$$M_n = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

Then:

- Find $E[M_n]$ and $Var(M_n)$ as a function of n.
- Using Chebyshev's inequality, find an upper bound on $P\left(|M_n - 1/2| \geq 1/100\right)$.