MTH202: Assignment 10

March 28, 2019

1. Suppose that X and Y are jointly distributed continuous random variables with joint density function given by:

$$f_{XY}(x,y) = \begin{cases} Ce^{x+y} & \text{for } x, y \in (-\infty, 0] \\ 0 & \text{otherwise} \end{cases}$$

   Answer the following questions:

   (a) Find C.

   (b) Compute $P(X < Y)$.

   (c) What are the marginal densities of X and Y?

   (d) Are X and Y independent?

   (e) Compute $E[X+Y], E[X], E[Y]$ and $E[XY]$

2. Two identical coins are flipped simultaneously. Let X be the number of HEADS and Y be the number of TAILS shown. What is the joint probability mass function of X and Y. What are the marginals?

3. Let X and Y (discrete random variables) have the joint mass function

$$P_{XY}(m,n) = \begin{cases} \frac{1}{2^{m+1}} & \text{for } m \geq n \\ 0 & \text{for } m < n \end{cases}$$

   Compute the marginals P_X and P_Y.

4. Let X, Y be two continuous random variables with joint probability density

$$f_{XY}(x,y) = \begin{cases} \frac{3}{4}(2x - x^2)e^{-y} & \text{for } 0 < x < 2, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

   Compute the following:

   (a) $P(X \geq 1, Y < 4)$.

   (b) $P(Y \geq 10)$.

   (c) $P(X < 5)$.

(d) $P(X \geq -1)$.

(e) $P(X \leq 2, Y < 5)$.

5. Let X and Y be two independent random variables. Find the probability mass function or the density function of $Z = X + Y$ in following cases:

   (a) $X \sim Bin(m, p)$, $Y \sim Bin(n, p)$.

   (b) $X \sim Poi(\lambda)$,$Y \sim Poi(\mu)$.

   (c) $X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$.

   (d) $X \sim \mathcal{N}(0, \sigma^2), Y \sim \mathcal{N}(0, \mu^2)$.

6. Let X and Y be real valued random variables. Show that:

   (a) $E[XY]^2 \leq E[X^2]E[Y^2]$.

   (b) $\sqrt{E[(X+Y)^2]} \leq \sqrt{E[X^2]} + \sqrt{E[Y^2]}$.

7. Let X, Y be jointly continuous random variables. Define $U = X + Y$ and $V = X - Y$.

   (a) Compute $E[U], E[V], Var(U), Var(V)$ in terms of $E[X], E[Y]$ and $E[XY]$.

   (b) If X and Y are independent, then are U, V independent as well?

   (c) If X and Y are independent, then are U, V uncorrelated?

8. Define covariance of X and Y as: $Cov(X, Y) = E[XY] - E[X]E[Y]$. Argue that:

   (a) $Cov(X, Y) = Cov(Y, X)$.

   (b) $Cov(X, X) = Var(X)$.

   (c) $Cov(aX, Y) = aCov(X, Y)$ for any constant $a \in \mathbb{R}$.