INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH MOHALI
MIDTERM 1
MTH 201- CURVES AND SURFACES

Date: 11/9/2019
Time: 1 Hour

Important Instructions.

(1) Answer all questions. Total points=20.
(2) *Mention clearly all the results you are using to answer any particular question.*
(3) *Use blue or black ink pen only to write you answers.*
(4) *Draw figures if needed to answer any question.*
(5) *Students are suggested not to cross or erase any answer which may be potentially wrong. There are partial credits for most of the questions. Writing some rough ideas is better than writing nothing.*

Questions:

(1) Suppose $a > 0, b > 0$ are constants. Consider the curve $\alpha(t) = (a\cos t, a\sin t, bt)$. It is a helix lying on the cylinder $x^2 + y^2 = a^2$.
(*i*) Find $\alpha'(t)$, $\frac{ds}{dt}$ and $\vec{T}$.
(*ii*) With respect to the base point $t = 0$ find the arc length function for α. Find the corresponding arc length parametrization.
(*iii*) Write down the definition of curvature of a parametrized curve. Compute the curvature of the above curve at an arbitrary point. $(3 + 2 + 5)$

(2) (*i*) Write down the definition of a surface.
(*ii*) Consider the set of points in $\mathbb{R}^3$ satisfying $y = x^2$. Draw a rough sketch of it. Show that it is surface. $(2 + 5)$

(3) Consider the curve $\alpha : [-1, 1] \to \mathbb{R}^3$ given by $\alpha(t) = (t, t^5, t^9)$. Show that the whole curve cannot be on a single plane. (3)

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Answers

Qn 1. (i) $\alpha(t) = (a\cos t, a\sin t, bt)$

$\Rightarrow \alpha'(t) = (-a\sin t, a\cos t, b)$ since a, b are constants.

$\Rightarrow \|\alpha'(t)\| = \sqrt{a^2(\sin^2 t + \cos^2 t) + b^2}$

$= \sqrt{a^2+b^2}$

Hence, $\frac{ds}{dt} = \|\alpha'(t)\| = \sqrt{a^2+b^2}$

and $\vec{T} = \frac{1}{\|\alpha'(t)\|} \cdot \alpha'(t) = \frac{1}{\sqrt{a^2+b^2}}(-a\sin t, a\cos t, b)$

(ii). $s = \int_0^t \|\alpha'(u)\|\, du = \int_0^t \sqrt{a^2+b^2}\, du = \sqrt{a^2+b^2}\cdot t$

since a, b are constants.

This is the arc length function.

. Hence, the arc length parametrization

is $\beta(s) = \alpha(t) = \alpha\left(\frac{1}{\sqrt{a^2+b^2}} s\right)$

$= \left(a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}}\right)$. Ans.

(iii) • Curvature of a smooth, regular, parametrized curve is defined to

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\|$$

• $\vec{T} = \frac{1}{\sqrt{a^2+b^2}} (-a \sin t, a \cos t, b)$

$$\Rightarrow \frac{d\vec{T}}{dt} = \frac{ds}{dt} \frac{d\vec{T}}{ds} \cdot \cancel{\frac{ds}{dt}} = \frac{1}{\sqrt{a^2+b^2}} (-a \cos t, -a \sin t, 0)$$

$$\Rightarrow \frac{d\vec{T}}{ds} = \frac{1}{a^2+b^2} (-a \cos t, a \sin t, 0), \text{ since } \frac{ds}{dt} = \sqrt{a^2+b^2}$$

$$\Rightarrow \kappa = \left\| \frac{d\vec{T}}{ds} \right\| = \frac{1}{a^2+b^2} \| (-a \cos t, a \sin t, 0) \|$$

$$= \frac{a}{a^2+b^2} \text{ since } a > 0.$$

Alternative :

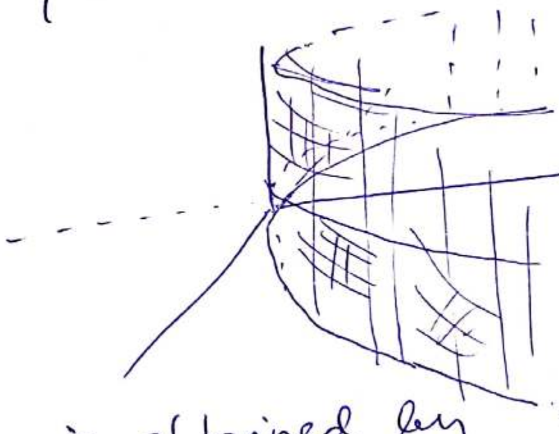
Use the formula $\kappa = \dfrac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3}$.

Qn 2. (i) A subset $S \subseteq \mathbb{R}^3$ is called a surface if $\forall p \in S$ there is an open set $\mathcal{U} \subseteq S$, $p \in \mathcal{U}$ and an open set $V \subseteq \mathbb{R}^2$ and a homeomorphism $\phi : \mathcal{U} \to V$.

Alternative :

A subset $S \subseteq \mathbb{R}^3$ .... if $\forall p \in S$ there is an open set $\mathcal{U} \subseteq S$, $p \in \mathcal{U}$ and a homeomorphism $\phi : \mathcal{U} \to \mathbb{D}$ where $\mathbb{D}$ is the open unit disk $= \{(x,y) : x^2+y^2<1\} \subseteq \mathbb{R}^2$.

(ii) .



The set is obtained by translating the parabola $y = x^2$ in the xy-plane vertically upward and downward

. Claim: $S = \{(x,y,z) : y = x^2\}$ is a surface.

Consider the map $\phi : S \to \mathbb{R}^2$ ~~area~~

$(x,y,z) \mapsto (x,z)$

~~Homeomorphism~~

It is the restriction of the projection from

$\mathbb{R}^3$ to the xz-plane.

Thus it is continuous.

- ϕ is <u>injective</u>: $\phi(x_1, y_1, z_1) = \phi(x_2, y_2, z_2)$

$$\Rightarrow (x_1, z_1) = (x_2, z_2)$$

$$\left.\begin{array}{l} \Rightarrow \quad x_1 = x_2 \\ \qquad z_1 = z_2 \\ \text{and } y_1 = x_1^2 = x_2^2 = y_2 . \end{array}\right\} \Rightarrow (x_1, y_1, z_1) = (x_2, y_2, z_2)$$

- ϕ is <u>surjective</u>:

Given any $(a, b) \in \mathbb{R}^2$, clearly $(a, a^2, b) \in S$ and $\phi(a, a^2, b) = (a, b)$.

- $\phi^{-1}(a, b) = (a, a^2, b)$ or

$$\phi^{-1}(x, y) = (x, x^2, y)$$

<u>Note</u>: The map $\begin{array}{l} \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \\ (x, y) \longmapsto (x, x^2, y) \end{array}$ is continuous since all the co-ordinate maps are. Hence, ϕ^{-1} is continuous.

Hence, $\forall p \in S$, take $\mathcal{U} = S$, $V = \mathbb{R}^2$ and $\phi : \mathcal{U} \to V$ as above. Then by the above definition 1 we are done.

(3) Any plane in $\mathbb{R}^3$ has an equation (5)
of the form
$$ax + by + cz = d$$
where $(a, b, c) \neq (0,0,0)$ and d are constants.
Hence, if α is contained in a plane then there are a, b, c, d with

$$at + bt^5 + ct^9 = d$$

for all $t \in [-1, 1]$.
But this is a polynomial in t whence it can have at most ~~one~~ finitely many root/s.
This contradiction proves the assertion.

<u>An Alternative</u>:
Consider the points $\alpha(0), \alpha(1), \alpha(1/2)$.
Find the equation of the plane containing them. Then show that $\alpha(t)$ cannot be in this plane for some $t \in [-1, 1]$.
If $P = \alpha(0) = (0,0,0)$, $Q = \alpha(1) = (1,1,1)$ and $R = \alpha(1/2) = (1/2, 1/2^5, 1/2^9)$ then $\vec{PQ} \times \vec{PR}$ is normal to the plane. The plane is passing through P. Hence the equation of the plane is $(x, y, z) \cdot (\vec{PQ} \times \vec{PR}) = 0$ etc.