HW 7

1. Can you think of two curves $\alpha, \beta : \mathbb{R} \to \mathbb{R}^2 \subseteq \mathbb{R}^3$ such that α cannot be taken to β by any rigid motion of $\mathbb{R}^3$ although α, β have the same curvature function $\kappa : \mathbb{R} \to \mathbb{R}$? Give an intuitive idea.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as follows:

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Show that f is discontinuous at $(0,0)$ but f has directional derivatives along any vector $v = (a,b) \in \mathbb{R}^2$ at $p = (0,0)$.

3. Show that f is $C^1 \Rightarrow f$ is C^0 where $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$, $\mathcal{U}$ open in $\mathbb{R}^n$. (See a reference if needed.)

4. Find directional derivatives of the functions f given below at the point along the vector v.

(i) $f(x,y,z) = x^2 + yz + xz^2$, $p = (1,1,1)$, $v = (1,2,3)$.

(ii) $f(x,y,z) = xy + yz + zx$, $p = (1,0,1)$, $v = (0,1,1)$.

(iii) $f(x,y) = xy(x+y)$, $p = (1,2)$, $v = (1,1)$.

5. For a smooth function $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$ show that $f'(p; v) = \text{grad}(f)(p) \cdot v$

$\forall\, p \in \mathcal{U}$ and $v \in \mathbb{R}^n$.

6. For a smooth function $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $p \in \mathcal{U}$ and a smooth curve $\alpha : (-\varepsilon, \varepsilon) \to \mathcal{U}$ with $\alpha(0) = p$ show that

$$(f \circ \alpha)'(0) = Jf(\alpha(0)) \cdot \alpha'(0).$$

using the chain rule.

7. Check if the following are diffeomorphisms onto the image
(i) $f : \mathbb{R} \to \mathbb{R}$ $\quad x \longmapsto e^x$
(ii) $f : \mathbb{R} \to \mathbb{R}$ $\quad x \longmapsto \dfrac{x}{x^2+1}$
(iii) $f : \mathbb{R} \to \mathbb{R}$ $\quad x \longmapsto x^3$
(iv) $f : \{(x,y) \in \mathbb{R}^2 : x > 0, -\pi < y < \pi\} \to \mathbb{R}^2$ $\quad (x,y) \longmapsto (x\cos y, x\sin y)$

8. Check if the following are allowable surface patches: First check if they are surface patches.
(i) $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ $\quad (x,y) \longmapsto (x, y, xy)$
(ii) $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ $\quad (x,y) \longmapsto (x, y^2, y^3)$
(iii) $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ $\quad (x,y) \longmapsto (x+x^2, y, y^2)$
(iv) $\varphi : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^3$ $\quad (x,y) \longmapsto (x, y, \sqrt{x^2+y^2})$
(v) $\varphi : (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$ $\quad (x,y) \longmapsto (\cos x, \sin x, y)$.
(vi) $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ $\quad (x,y) \longmapsto (x^2, xy, y^2)$