Solution to HW7

1) Try yourself.

2) Fix $m \neq 0$. Then $\lim_{y \to 0} f(my^2, y) = \frac{m}{1+m^2}$

Thus f is not continuous at $(0,0)$

Given $v = (a, b)$

$$f'((0,0); v) = \lim_{t \to 0} \frac{f(tv) - f(0)}{t}$$

$$= \lim_{t \to 0} \frac{f(ta, tb)}{t} = \lim_{t \to 0} \frac{t^3 ab^2}{t(t^2a^2 + t^4 b^4)}$$

$$= \lim_{t \to 0} \frac{ab^2}{a^2 + tb^4} = \begin{cases} \frac{b^2}{a} & \text{if } a \neq 0 \\ 0 & \text{if } a = 0 \end{cases}$$

3) See below.

4) (i) $f(p+tv) = f(1+t, 1+2t, 1+3t)$

$= (1+t)^2 + (1+2t)(1+3t) + (1+t)(1+3t)^2$

$= 3 + 2t + 5t + 7t +$ higher order terms

$f(p) = 3$

Hence, $f'(p; v) = \lim_{t \to 0} \frac{f(p+tv) - f(p)}{t} = 2+5+7 = 14$

(ii) $\text{grad} f = (f_x, f_y, f_z) = (y+z, x+z, x+y)$

$\Rightarrow \text{grad}(f)(p) = (1, 2, 1)$

$\Rightarrow f'(p; v) = \text{grad} f(p) \cdot v = (1,2,1) \cdot (0,1,1) = 3$

(iii) $\text{grad} f = (2xy + y^2, x^2 + 2xy) \Rightarrow \text{grad} f(p) = (8, 5)$

$\Rightarrow f'(p; v) = (8,5) \cdot (1,1) = 13$

5) See below.

6) Recall: If $f(x) = (f_1(x), \cdots, f_m(x)) : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is smooth then

$$Jf = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Thus when $n = 1$, $Jf = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_m}{\partial x_1} \end{pmatrix}$

Therefore, if $\alpha : (-\varepsilon, \varepsilon) \longrightarrow U \subseteq \mathbb{R}^n$ is a curve, $\alpha(t) = (\alpha_1(t), \cdots, \alpha_n(t))$ then

$$J\alpha(0) = \begin{pmatrix} \alpha_1'(0) \\ \vdots \\ \alpha_n'(0) \end{pmatrix} = \alpha'(0) \text{ written as a column vector.}$$

The rest is chain rule.

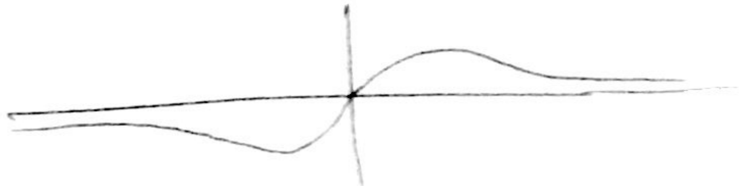
7) (i) $f'(x) = e^x > 0$ Hence, we are done by the inverse function theorem. Here $\mathrm{Im}(f) = (0, \infty)$. Clearly we can directly prove that $\log : (0, \infty) \to \mathbb{R}$ is a smooth inverse of e^x.

(ii) One can directly find f^{-1} ~~inverse~~ and note that f, f^{-1} are smooth. Or just find $f'(x)$ and show that it is nonzero $\forall x \in \mathbb{R}$ if f were to be a diffeomorphism onto image.

$$f'(x) = \frac{(x^2+1) - 2x \cdot x}{(x^2+1)^2} = \frac{1 - x^2}{(x^2+1)^2}$$

Since $f'(\pm 1) = 0$ f cannot be a diffeomorphism onto its image.

Ex: Check if f is a homeomorphism onto its image.



(iii) $f'(0)=0$ Hence f cannot be a diffeomorphism onto image. Note: $f^{-1}(x) = x^{1/3}$ is continuous. Hence, f is a homeomorphism $\mathbb{R} \to \mathbb{R}$.

(iv) $$Jf = \begin{pmatrix} \cos y & -x\sin y \\ \sin y & x\cos y \end{pmatrix}$$

$$\Rightarrow \det Jf = x > 0$$

Hence, by the inverse function theorem we are done.

Note: This is just transformation of polar co-ordinates into Cartesian co-ordinates.

8. (i) $\varphi_x = (1,0,y)$, $\varphi_y = (0,1,x)$

$$\Rightarrow \varphi_x \times \varphi_y = \begin{vmatrix} i & j & k \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = -y\vec{i} - x\vec{j} + \vec{k} \neq \vec{0} \quad \forall (x,y)$$

Hence, φ is an allowable surface patch.

Well one needs to check that φ is a homeomorphism onto its image $S = \{(x,y,z) : z = xy\}$

Clearly, φ is bijective onto S and $\varphi^{-1}(x,y,z) = (x,y)$ which is continuous.

(ii) $\varphi_y = (0, 2y, 3y^2) \Rightarrow \varphi_y(0,0) = 0$.

Hence, φ cannot be an allowable surface patch.

Check if it is a surface patch. (4)

(iii) Clearly φ is ~~not onto image~~ not even 1-1 : $\varphi(0,y) = \varphi(-1,y)$.

Hence, φ is not a surface patch.

(iv) • φ is a surface patch for the cone $z=\sqrt{x^2+y^2}$ with vertex $= (0,0,0)$ removed.

• φ is smooth.

• $\varphi_x = \left(1, 0, \frac{x}{\sqrt{x^2+y^2}}\right)$, $\varphi_y = \left(0, 1, \frac{y}{\sqrt{x^2+y^2}}\right)$

Clearly $\varphi_x \times \varphi_y \neq \vec{0}$ $\forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

Hence, φ is an allowable surface patch.

(v) • φ is a surface patch for the cylider $x^2+y^2=1$ with one line : $x=1, y=0$ removed.

• φ is smooth

• $\varphi_x = (-\sin x, \cos x, 0)$

$\varphi_y = (0, 0, 1)$

$\Rightarrow \varphi_x \times \varphi_y = \sin x\, \vec{j} + \cos x\, \vec{i} \neq 0 \quad \forall (x,y)$

$\Rightarrow \varphi$ is an allowable surface patch.

(3) and (5) : The crucial thing to use is the Mean Value Theorem: If $f : [a,b] \to \mathbb{R}$ is C^1 then $\frac{f(b)-f(a)}{b-a} = f'(c)$ for some $c \in [a,b)$.

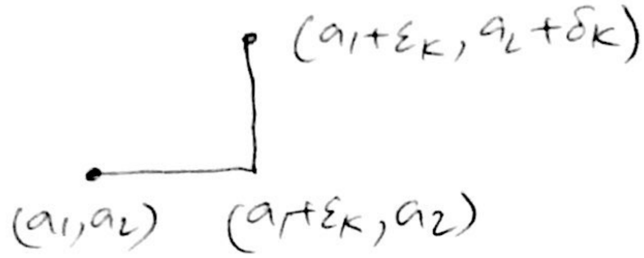
For (3) suppose $\{x_k\}$ is a sequence in $\mathcal{U}$ converging to x where

$$x_k = (a_{1k}, a_{2k}, \dots, a_{nk}) \quad \text{and}$$
$$x = (a_1, a_2, \dots a_n).$$

For simplicity assume $n = 2$.

Write
$$a_{1k} = a_1 + \varepsilon_k$$
$$a_{2k} = a_2 + \delta_k$$

We have $\varepsilon_k \to 0$, $\delta_k \to 0$ and we want to show that $f(a_1+\varepsilon_k, a_2+\delta_k) \to f(a_1, a_2)$.



Now, $f(a_1+\varepsilon_k, a_2+\delta_k) - f(a_1, a_2)$

$$= \left[f(a_1+\varepsilon_k, a_2+\delta_k) - f(a_1+\varepsilon_k, a_2)\right] + \left[f(a_1+\varepsilon_k, a_2) - f(a_1, a_2)\right]$$

$$= f_y(a_1+\varepsilon_k, a_2+\delta_k') \cdot \delta_k + f_x(a_1+\varepsilon_k', a_2) \cdot \varepsilon_k$$

for some $0 < \delta_k' < \delta_k$ and $0 < \varepsilon_k' < \varepsilon_k$

Now, as f_x, f_y are continuous
$$f_y(a_1+\varepsilon_k, a_2+\delta_k') \to f_y(a_1, a_2) \quad \text{and}$$
$$f_x(a_1+\varepsilon_k', a_2) \to f_x(a_1, a_2) \quad \text{as } \varepsilon_k \to 0,\ \delta_k \to 0$$

Hence, $\lim_{k\to\infty} \left[f(a_1+\varepsilon_k, a_2+\delta_k) - f(a_1, a_2)\right] = 0.$

Do (5) similarly.