

## Solution to HW6

1) Suppose  $\alpha: I \rightarrow \mathbb{R}^2$  is such a curve.  
Then  $\kappa_s$  is also constant. Therefore,  
if  $\varphi$  is the turning angle we have

$$\frac{d\varphi}{ds} = \text{constant} = c \text{ say.}$$

Case 1:  $c = 0$

Then  $\varphi$  is constant

$\Rightarrow \vec{T} = (\cos \varphi(s), \sin \varphi(s))$  is a constant vector.

If  $\vec{T} = (a_1, a_2)$  then

$\alpha(s) = (a_1 s + b_1, a_2 s + b_2)$  for some

constants  $b_1, b_2$ .

Note:  $(a_1, a_2) \neq (0, 0)$  ~~since these are~~  
since  $a_1^2 + a_2^2 = 1$ .

Clearly  $\alpha$  is contained in the plane

$$a_2(x - b_1) = a_1(y - b_2).$$

Case 2:  $c \neq 0$

$\varphi = cs + c_0$  for some constant  $c_0$

$$\Rightarrow \vec{T}(s) = (\cos(cs + c_0), \sin(cs + c_0))$$

$$\Rightarrow \alpha(s) = \left( \frac{1}{c} \sin(cs + c_0), -\frac{1}{c} \cos(cs + c_0) \right) + (c_1, c_2)$$

for some constants  $c_1, c_2$

Clearly  $\alpha$  is contained in the circle

$$(x - c_1)^2 + (y - c_2)^2 = \frac{1}{c^2}$$

$$2) x_p = \cos s \Rightarrow \frac{d\varphi}{ds} = \cos s \quad (2)$$

$\Rightarrow \varphi(s) = \sin s + c_0$  for some constant  $c_0$

$$\Rightarrow \vec{T}(s) = (\cos(\sin s + c_0), \sin(\sin s + c_0))$$

$P = (0,0)$  is on the curve. If length is measured from  $P$  then

$$\vec{T}(0) = (\cos c_0, \sin c_0) = (1,0)$$

Hence, we can take  $c_0 = 0$ .

$$\Rightarrow \vec{T}(s) = (\cos(\sin s), \sin(\sin s))$$

$$\Rightarrow \alpha(s) = \left( \int_0^s \cos(\sin u) du, \int_0^s \sin(\sin u) du \right)$$

(i)  $\Rightarrow$  (ii):

$$3) \text{ Let } e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(\*)  $Ae_i = i^{\text{th}}$  column vector of  $A$ .

From  $AA^t = I$  deduce that  $\{Ae_1, Ae_2, Ae_3\}$  is an orthonormal basis of  $\mathbb{R}^3$ .

Given  $v, w \in \mathbb{R}^3$  write  $v = a_1 e_1 + a_2 e_2 + a_3 e_3$ ,

$w = b_1 e_1 + b_2 e_2 + b_3 e_3$ ,  $a_i, b_i \in \mathbb{R}$  and then use that

$v \mapsto Av$  is a linear map.

(ii)  $\Rightarrow$  (i): Apply (ii) for  $v, w \in \{e_1, e_2, e_3\}$ . Then use (\*).

(ii)  $\Rightarrow$  (iii): Immediate

(iii)  $\Rightarrow$  (ii): Compute  $A(v+w)$ ,  $A(v+w)$

4. (i) check it for  $v, w \in \{e_1, e_2, e_3\}$  or  $\{\vec{i}, \vec{j}, \vec{k}\}$  and then use linearity.

(ii) Do it yourself.



$$5. (i) \quad \alpha(t) = (t, t^2, t^3) \quad (3)$$

$$\Rightarrow \left. \begin{aligned} \alpha'(t) &= (1, 2t, 3t^2) \\ \alpha''(t) &= (0, 2, 6t) \\ \alpha'''(t) &= (0, 0, 6) \end{aligned} \right\} \Rightarrow \left. \begin{aligned} \alpha'(0) &= (1, 0, 0) = \vec{i} \\ \alpha''(0) &= (0, 2, 0) = 2\vec{j} \\ \alpha'''(0) &= (0, 0, 6) = 6\vec{k} \end{aligned} \right\}$$

clearly  $\vec{T}(0) = \vec{i}$

$\vec{b}$  is parallel to  $\alpha'(t) \times \alpha''(t)$ .

$$\Rightarrow \vec{b}(0) = \frac{\vec{i} \times 2\vec{j}}{\|\vec{i} \times 2\vec{j}\|} = \vec{k}$$

$$\Rightarrow \vec{n}(0) = \vec{b}(0) \times \vec{T}(0) = \vec{k} \times \vec{i} = \vec{j}$$

$$\kappa(0) = \frac{\|\alpha'(0) \times \alpha''(0)\|}{\|\alpha'(0)\|^3} = 2$$

$$\tau(0) = \frac{(\alpha'(0) \times \alpha''(0)) \cdot \alpha'''(0)}{\|\alpha'(0) \times \alpha''(0)\|^2} = \frac{2\vec{k} \cdot 6\vec{k}}{\|2\vec{k}\|^2} = 3$$

$$(ii) \quad \alpha(t) = (t \cos t, t \sin t, t)$$

$$\Rightarrow \alpha'(t) = (\cos t - t \sin t, \sin t + t \cos t, 1)$$

$$\alpha''(t) = (-2 \sin t - t \cos t, 2 \cos t - t \sin t, 0)$$

$$\alpha'''(t) = (-3 \cos t + t \sin t, -3 \sin t - t \cos t, 0)$$

$$\Rightarrow \alpha'(0) = (1, 0, 1) = \vec{i} + \vec{k}$$

$$\left. \begin{aligned} \alpha''(0) &= (0, 2, 0) \\ &= 2\vec{j} \end{aligned} \right\}, \alpha'''(0) = (-3, 0, 0) = -3\vec{i} \text{ etc.}$$

6. (i) Yes, This curve is on the plane  $z = x + y + 1$ .

(ii) No. Calculate torsion and show that it is not zero. OR use that  $e$  is a transcendental number