Solution to HW5

2) If $\vec{T} = (\cos\varphi(s), \sin\varphi(s))$ then $\kappa_s = \frac{d\varphi}{ds}$ and

(i) $n_s = (-\sin\varphi(s), \cos(\varphi(s)))$.

$$\Rightarrow \frac{dn_s}{ds} = -\varphi'(s)(\cos\varphi(s), \sin\varphi(s)) = -\kappa_s\vec{T}.$$

(ii) Since φ is smooth, κ_s is smooth.

3) Signed Curvature of $\beta = \dfrac{\vec{k}\cdot(\beta'(t)\times\beta''(t))}{\|\beta'(t)\|^3}$

$$= \frac{\vec{k}\cdot(-\alpha'(a+b-t)\times\alpha''(a+b-t))}{\|-\alpha'(a+b-t)\|} = -\text{Signed}$$

curvature of α at $(a+b-t)$

Note: $\beta(a) = \alpha(b)$, $\beta(b) = \alpha(a)$ β is the "same" curve traversed in the opposite direction.
Roughly speaking signed curvature depends not only on the "shape" of the curve but also on the direction.

4) $x = a\cos t$, $y = b\sin t$ is a parametrization of the ellipse.

Here, $\alpha(t) = (a\cos t, b\sin t)$

$$\Rightarrow \alpha'(t) = (-a\sin t, b\cos t),$$

$$\alpha''(t) = -(a\cos t, b\sin t)$$

$$\Rightarrow \alpha'\times\alpha'' = ab\vec{k}$$

$$\Rightarrow \kappa = \frac{\|\alpha'\times\alpha''\|}{\|\alpha'\|^3} = \frac{ab}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}$$

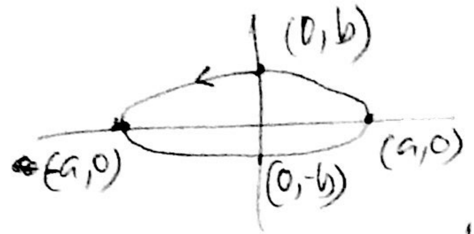
$$= \frac{ab}{\sqrt{(a^2-b^2)\sin^2 t + b^2}}$$

Hence κ is maximum for $\sin t = 0$ and minimum for $\sin t = \pm 1$

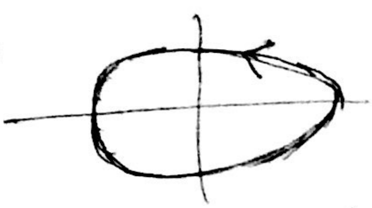
In $[0, 2\pi]$, $\sin t = 0 \Leftrightarrow t = 0, \pi, 2\pi$ (2)

and $\sin t = \pm 1 \Leftrightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}$

It follows that curvature is maximum at the points $(\pm a, 0)$ and minimum at $(0, \pm b)$



5) (i)



$$\kappa = \frac{|r^2 + 2r'^2 - rr''|}{(r^2 + r'^2)^{3/2}}$$

$r = 2 + \cos\theta \Rightarrow r' = -\sin\theta,\ r'' = -\cos\theta$

$\Rightarrow r^2 + r'^2 = (2+\cos\theta)^2 + \sin^2\theta = 4 + 3\cos\theta$

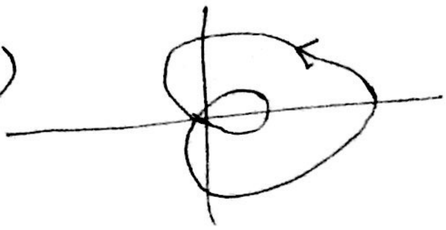
$r^2 + 2r'^2 - rr'' = (2+\cos\theta)^2 + 2\sin^2\theta + \cos\theta(2+\cos\theta)$

$= 4 + 4\cos\theta + \cos^2\theta + 2\sin^2\theta + 4\cos\theta + \cos^2\theta$

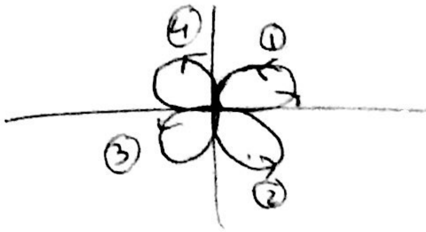
$= 6 + 8\cos\theta$

Hence $\kappa = \dfrac{|6 + 8\cos\theta|}{(4 + 3\cos\theta)^{3/2}}$

(ii)



(iii)



The arrows show the direction as θ increases from 0 to 2π.

6) Apply $\kappa = \dfrac{|y''|}{(1+y'^2)^{3/2}}$