

Solution to HW 4

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1. (i) The function $\sqrt{1-t^2}$ is not smooth on $[0,1]$
i.e. we cannot find an open interval $(a,b) \supseteq [0,1]$
and a smooth function $f(t)$ on (a,b) such
that $f|_{[0,1]} = \sqrt{1-t^2}$. Suppose we could.

Then $f'(t) = \frac{d}{dt}(\sqrt{1-t^2}) = \frac{-t}{\sqrt{1-t^2}} \quad \forall t \in (0,1)$.

$\Rightarrow \lim_{t \rightarrow 1} f'(t) = f'(1) = \lim_{t \rightarrow 1} \frac{-t}{\sqrt{1-t^2}}$ since f is smooth.

However, the right hand limit does not exist.

(ii) β is not smoothly extendable beyond 0.

The proof is similar to that of 1(i).

2. (i) Book's definition: $S \subseteq \mathbb{R}^3$ is called a
surface if $\forall p \in S \exists$ an open neighborhood
 V of p in S and a homeomorphism from V to
an open subset of \mathbb{R}^2 .

Clearly ~~our definition~~ implies if S is a surface
by our definition then it is so as per the
book's definition.

~~Suppose~~ Let S be a surface ^{by} the book's definition. ~~the~~
Let $p \in S$. Then there is an open set $V \subseteq S$,
 $p \in V$ and a homeomorphism $\varphi: V \rightarrow U \subseteq \mathbb{R}^2$
where $U \subseteq \mathbb{R}^2$ is open. Let $q = \varphi(p) \in U$.
Since $q \in U$ and U is open there is λ disc
 $B \subseteq U$ with $q \in B$. Let $W = \varphi^{-1}(B)$. One just
checks that $\varphi: \varphi^{-1}(B) \rightarrow B$ is a homeomorphism.

2. (ii) Suppose S is a surface. We will use the book's definition. Let $V \subseteq S$ be open. Let $p \in V$. Then there is an open set $W \subseteq S$ and a homeomorphism $\varphi: V \rightarrow U$ where $U \subseteq \mathbb{R}^2$ is open.

Now, check V, W open in $S \Rightarrow V \cap W$ is open in S . Then check that $\varphi(V \cap W)$ is open in U and hence in \mathbb{R}^2 . Finally $\varphi: V \cap W \rightarrow \varphi(V \cap W)$ is a homeomorphism.

3. Let $\gamma: J \rightarrow I$ be the inverse of φ . Then $\varphi \circ \gamma: J \rightarrow J$ is the identity map and $\gamma \circ \varphi: I \rightarrow I$ is the identity map.

In particular $\gamma \circ \varphi(t) = t \quad \forall t \in I$. Take derivative and apply chain rule.

$$\gamma'(\varphi(t)) \cdot \varphi'(t) = 1$$

$$\Rightarrow \varphi'(t) \neq 0.$$

4. a) $\alpha(t) = t(\cos t, \sin t)$
 $\Rightarrow \alpha'(t) = (\cos t, \sin t) + t(-\sin t, \cos t)$

$$\Rightarrow \|\alpha'(t)\| = \sqrt{1+t^2} > 0 \quad \forall t$$

Hence $\alpha'(t) \neq 0$.

Thus α is regular with speed $\|\alpha'(t)\| = \sqrt{1+t^2}$. Unit tangent vector = $\frac{1}{\|\alpha'(t)\|} \alpha'(t)$

$$= \left(\frac{\cos t - t \sin t}{\sqrt{1+t^2}}, \frac{\sin t + t \cos t}{\sqrt{1+t^2}} \right)$$

$$4.6) \quad \alpha(t) = (t - \sin t, 1 - \cos t)$$

(3)

$$\Rightarrow \alpha'(t) = (1 + \cos t, \sin t)$$

$$\begin{aligned} \Rightarrow \|\alpha'(t)\| &= \sqrt{(1 + \cos t)^2 + \sin^2 t} \\ &= \sqrt{2 + 2\cos t} = \sqrt{2} \sqrt{1 + \cos t} \end{aligned}$$

Clearly $\alpha'(\pi) = 0$. Hence α is not regular

and its speed = $\sqrt{2} \cdot \sqrt{1 + \cos t} = 2\sqrt{2} |\cos t/2|$.

Its unit tangent vector, when defined is

$$\frac{1}{\|\alpha'(t)\|} \alpha'(t) = \left(\frac{1 + \cos t}{2\sqrt{2} |\cos t/2|}, \frac{\sin t}{2\sqrt{2} |\cos t/2|} \right)$$

$$c) \quad \alpha(t) = e^{kt} (\cos t, \sin t)$$

$$\Rightarrow \alpha'(t) = k e^{kt} (\cos t, \sin t) + e^{kt} (-\sin t, \cos t) \quad (k \text{ is constant})$$

$$\Rightarrow \|\alpha'(t)\| = \sqrt{1+k^2} e^{kt}$$

Thus $\alpha'(t) \neq 0 \quad \forall t$.

Hence α is regular with speed $\sqrt{1+k^2} e^{kt}$ at time t and unit tangent vector =

$$\frac{1}{\|\alpha'(t)\|} \alpha'(t) = \left(\frac{k \cos t - \sin t}{\sqrt{1+k^2}}, \frac{k \sin t + \cos t}{\sqrt{1+k^2}} \right)$$

$$e) \quad \alpha(t) = (t^2, t^2+1, t^2+2)$$

$$\Rightarrow \alpha'(t) = 2t (1, 1, 1)$$

$$\Rightarrow \|\alpha'(t)\| = 2\sqrt{3} \cdot t, \quad \forall t \in (0, \infty)$$

Thus α is regular, speed $2\sqrt{3} t$, unit tangent vector $\frac{1}{\sqrt{3}} (1, 1, 1)$.

Note: α traces a straight line.

5. (c) $d(t) = e^{kt} (\cos t, \sin t)$ (4)

$t_0 = 0$

$$s = \int_0^t \|d'(t)\| dt = \int_0^t \sqrt{1+k^2} e^{kt} dt$$

$$= \frac{\sqrt{1+k^2}}{k} (e^{kt} - 1)$$

$$\Rightarrow t = \frac{1}{k} \log \left(1 + \frac{ks}{\sqrt{1+k^2}} \right)$$

Hence, the arc length parametrization is

$$\beta(s) = d(t) = d\left(\frac{1}{k} \log \left(1 + \frac{ks}{\sqrt{1+k^2}} \right)\right)$$

$$= \left(1 + \frac{ks}{\sqrt{1+k^2}} \right) \left(\cos \frac{1}{k} \log \left(1 + \frac{ks}{\sqrt{1+k^2}} \right), \sin \frac{1}{k} \log \left(1 + \frac{ks}{\sqrt{1+k^2}} \right) \right)$$

(d) ✓

(e) $s = \int_1^t \|d'(u)\| du = \int_1^t 2\sqrt{3} u du$

$$= \left[\sqrt{3} u^2 \right]_1^t = \sqrt{3} t^2 - \sqrt{3}$$

$$\Rightarrow t^2 = \frac{1}{\sqrt{3}} s + 1$$

Hence, the arc length parametrization is

$$\begin{aligned} \beta(s) = d(t) &= (t^2, t^2+1, t^2+2) \\ &= \left(\frac{1}{\sqrt{3}} s + 1, \frac{1}{\sqrt{3}} s + 2, \frac{1}{\sqrt{3}} s + 3 \right) \end{aligned}$$

(It is clear that β traces a straight line.)

6. $d''(t) = (d_1''(t), d_2''(t), d_3''(t)) = 0$
 $\Rightarrow d_i''(t) = 0, i=1,2,3$ etc.