HW - 11

$*$ means difficult $**$ Too difficult

1. If $T = D\mathcal{G}_p$ then show that

$$T^2 + 2HT + K = 0$$

where H, K are respectively the mean and the Gaussian curvature.

2.* Show that the Gaussian curvature is positive at all points of an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

If $a > b > c$ find points of maximum and minimum curvature.

3.* Consider the circle $C: (y-2)^2 + z^2 = 1, x = 0$. Let S be the surface obtained by revolving C about the z-axis. Describe the points ~~at~~ which Gaussian curvature is positive, negative and zero.

4. Show that Gaussian curvature of

$S_1: x^2+y^2 = 1$, (Cylinder) $S_2: z = \sqrt{x^2+y^2}, z > 0$ (Cone)

are both zero at all points. What about mean curvature?

5. ~~Show that~~ Given a surface patch $\varphi: U \to S$ let $\vec{N} = \varphi_u \times \varphi_v / \|\varphi_u \times \varphi_v\|$, $p = \varphi(u,v)$.

1) Check (or recall) $\vec{N_u}, \vec{N_v} \in T_pS$

2) $K = 0$ at ~~point~~ p iff $\vec{N_u}(p), \vec{N_v}(p)$ are linearly dependent

3) In general $\vec{N_u} \times \vec{N_v} = K\, \varphi_u \times \varphi_v$

Thus $K > 0$ or < 0 according as $\vec{N_u} \times \vec{N_v}$ has direction as that of $\vec{N}$, and $K=0$ iff $\vec{N}_u \times \vec{N}_v = \vec{0}$.

6) (i) Suppose $S \subseteq \mathbb{R}^3$ is a surface and $S' \subseteq \mathbb{R}^3$ is obtained from S by applying a rigid motion f of $\mathbb{R}^3$. Show that curvature of S at $p \in S$ is that of S' at $f(p)$. (2)

(ii) Check the effect of the transformations

$f: \mathbb{R}^3 \to \mathbb{R}^3$ where $c>0$ is a constant,
$p \mapsto cp$

on curvature of surfaces in $\mathbb{R}^3$.

7)** Suppose $S \subseteq \mathbb{R}^3$ any (compact) surface and $p \in \mathbb{R}^3$, $p \notin S$. Suppose $p_0 \in S$ is such that p_0 is the farthest point of S from p. Then $K(p_0) \geq 0$.
Find an intuitive proof.

8) Apply the technique of finding curvature of a surface of revolution to find a surface such that $K=-1$ at all points.

Remark: Make yourself comfortable with the calculation of the 1st, 2nd fundamental forms, Gauss map, Gaussian/Mean curvature in surface patches – by doing examples. Here are some:

(i) $\varphi: \mathbb{R}^2 \to \mathbb{R}^3$, $\varphi(u,v) = (u, u^2, v)$

(ii) $\varphi: (0,2\pi)\times(0,2\pi) \to \mathbb{R}^3$, $\varphi(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)$

(iii) $\varphi: \mathbb{R}^2 \to \mathbb{R}^3$, $\varphi(u,v) = (u, v, u^3 - 3uv^2)$