Solution to HW-11

1. Fix a surface patch $\varphi : \mathcal{U} \to S$ around p. Then w.r.t. $\{\varphi_u, \varphi_v\}$ the Gauss map T has matrix

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = -\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

The characteristic ~~equation~~ polynomial of $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is

$$\det\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix} - \lambda I\right)$$

$$= \det\left(\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix} + \lambda I\right)$$

$$= \frac{1}{\det\begin{pmatrix} E & F \\ F & G \end{pmatrix}} \det\left(\begin{pmatrix} L & M \\ M & N \end{pmatrix} + \lambda \begin{pmatrix} E & F \\ F & G \end{pmatrix}\right)$$

$$= \frac{1}{\det\begin{pmatrix} E & F \\ F & G \end{pmatrix}} \det\begin{pmatrix} L+\lambda E & M+\lambda F \\ M+\lambda F & N+\lambda G \end{pmatrix}$$

$$= \frac{\lambda^2(EG-F^2) + (LG+NE-2MF)\lambda + LN-M^2}{EG-F^2}$$

$$= \lambda^2 + 2H\lambda + K$$ where H = mean curvature, K = Gaussian "

Thus by Cayley-Hamilton's theorem

$$T^2 + 2HT + K = 0.$$

2. The portion of the ellipsoid above the xy-plane is the graph of $f(x,y) = c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

$$f(u,v) = c\sqrt{1-\frac{u^2}{a^2}-\frac{v^2}{b^2}}$$

$$\Rightarrow f_u = -\frac{c}{a^2}\cdot\frac{u}{\sqrt{1-\frac{u^2}{a^2}-\frac{v^2}{b^2}}}, \quad f_v = -\frac{c}{b^2}\frac{v}{\sqrt{1-\frac{u^2}{a^2}-\frac{v^2}{b^2}}}$$

$$\Rightarrow f_{uu} = -\frac{c}{a^2}\left\{\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-1/2} + u\left(-\tfrac{1}{2}\right)\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-3/2}\cdot\left(\frac{-2u}{a^2}\right)\right\}$$

$$= -\frac{c}{a^2}\left\{\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-1/2} + \frac{u^2}{a^2}\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-3/2}\right\}$$

$$= -\frac{c}{a^2}\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-3/2}\left\{1-\frac{u^2}{a^2}-\frac{v^2}{b^2}+\frac{u^2}{a^2}\right\}$$

$$= -\frac{c}{a^2}\left(1-\frac{v^2}{b^2}\right)\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-3/2}$$

Similarly, $f_{vv} = -\frac{c}{b^2}\left(1-\frac{u^2}{a^2}\right)\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-3/2}$,

$$f_{uv} = -\frac{c}{a^2b^2}uv\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-3/2}$$

Now, apply the formula $K = \dfrac{f_{uu}f_{vv}-f_{uv}^2}{(1+f_u^2+f_v^2)^2}$.

$$\text{Numerator} = \frac{c^2}{a^2b^2}\left(1-\frac{u^2}{a^2}\right)\left(1-\frac{v^2}{b^2}\right)\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-3}$$

$$- \frac{c^2}{a^4b^4}u^2v^2\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-3}$$

$$= \frac{c^2}{a^2b^2}\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-3}\left\{\left(1-\frac{u^2}{a^2}\right)\left(1-\frac{v^2}{b^2}\right)-\frac{u^2v^2}{a^2b^2}\right\}$$

$$= \frac{c^2}{a^2b^2}\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-2}$$

$$\text{Denominator} = \left\{1+\frac{c^2}{a^4}\cdot\frac{u^2}{1-\frac{u^2}{a^2}-\frac{v^2}{b^2}}+\frac{c^2}{b^4}\frac{v^2}{1-\frac{u^2}{a^2}-\frac{v^2}{b^2}}\right\}^2$$

$$= a^{-8}b^{-8}\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-2}\left\{\cancel{a^4b^4}+c^2u^2+c^2v^2+a^4b^4\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)\right\}^2$$

$$= a^{-8}b^{-8}\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-2}\left\{c^2u^2+c^2v^2+a^4b^4-b^2u^2-a^2v^2\right\}^2$$

$$= a^{-8}b^{-8}\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-2}\left\{a^4b^4+(c^2-b^2)u^2+(c^2-a^2)v^2\right\}^2$$

$$= \cancel{a^{-8}b^{-8}}$$

$$\Rightarrow K = \frac{c^2a^6b^6}{\left(a^4b^4-(b^2-c^2)u^2-(a^2-c^2)v^2\right)^2} \longrightarrow (*)$$

Note: $\frac{u^2}{a^2}+\frac{v^2}{b^2} \leq 1$. By smoothness of K $(*)$ is valid for $\frac{u^2}{a^2}+\frac{v^2}{b^2}=1$ too.

Let $s=\frac{u^2}{a^2}$, $t=\frac{v^2}{b^2}$. Then $s+t\leq 1$ and $s,t\geq 0$.

The expression in the denominator of K

$$= \left(a^4b^4-(b^2-c^2)a^2s-(a^2-c^2)b^2t\right)^2$$

is minimum or maximum according as $a^4b^4-(b^2-c^2)a^2s-(a^2-c^2)b^2t$ is minimum or maximum subject to $s,t\geq 0$, $s+t\leq 1$

Since $(a^2-c^2)>0$, $b^2-c^2>0$ we can simply (4)
say that the denominator max or min
according as $(a^2-c^2)b^2t+(b^2-c^2)a^2s$ is min
or max resp.

Thus K is min or max according as
$(a^2-c^2)b^2t+(b^2-c^2)a^2s$ is min or max resp
subject to $s,t\geq 0$, $s+t\leq 1$.

Since $(a^2-c^2)b^2t+(b^2-c^2)a^2s\geq 0$ it follows
that min. happens for $s=t=0$ i.e.
$u,v=0$.

This is a linear programing problem. The maximum
occurs at a vertex because along every line
in the triangle $s+t\leq 1$, $s,t\geq 0$ the
function $(a^2-c^2)b^2t+(b^2-c^2)a^2s$ is monotonic.

The vertex $(0,0)$ gives minimum.
Look at $(0,1)$ and $(1,0)$ and compare:

$(s,t)=(0,1)$: $(a^2-c^2)b^2 = a^2b^2-b^2c^2$
$(s,t)=(1,0)$: $(b^2-c^2)a^2 = a^2b^2-a^2c^2$

Since $a>b>c$ the maximum is at
$(s,t)=(1,0)$ i.e. $u=\pm a$, $v=0$

Conclusion: On the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$,
$a>b>c>0$, $K>0$ at all points and K is max.
at $(\pm a,0,0)$ and min. at $(0,0,\pm c)$

Note: Reflection in the xy- plane gives an isometry from the upper half of the ellipsoid to the lower half. By Theorem Egregium or by an actual calculation we can calculate curvature on the lower half of the ellipsoid. It follows that $K(x,y,z) = K(x,y,-z)$. Hence, for K_{max}, K_{min} it is enough to look all $z \geq 0$.

3. Consider the parametrization

$$t \mapsto (0, 2+\cos t, \sin t) \text{ of } C$$

To get a surface patch we restrict t to $(0, 2\pi)$.

Now, consider $\varphi(\theta, t) = ((2+\cos t)\cos\theta, (2+\cos t)\sin\theta, \sin t)$

$t, \theta \in (0, 2\pi)$. This is a surface patch.

$$\varphi_\theta = (-(2+\cos t)\sin\theta, (2+\cos t)\cos\theta, 0)$$

$$\varphi_t = (-\sin t\cos\theta, -\sin t\sin\theta, \cos t)$$

$$\varphi_\theta \times \varphi_t = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -(2+\cos t)\sin\theta & \cdots & \\ -\sin t\cos\theta & -\sin t\sin\theta & \cos t \end{vmatrix}$$

$$= (2+\cos t)\cos t\cos\theta\,\vec{i} + (2+\cos t)\cos t\sin\theta\,\vec{j} + (2+\cos t)\sin t\,\vec{k}$$

Since $2+\cos t > 0$, $\vec{N} = (\cos t\cos\theta, \cos t\sin\theta, \sin t)$

$$\varphi_{\theta\theta} = \left(-(2+\cos t)\cos\theta,\ -(2+\cos t)\sin\theta,\ 0\right)$$

$$\varphi_{\theta t} = \left(\sin t\sin\theta,\ -\sin t\cos\theta,\ 0\right)$$

$$\varphi_{tt} = \left(-\cos t\cos\theta,\ -\cos t\sin\theta,\ -\sin t\right)$$

$$E = \varphi_\theta \cdot \varphi_\theta = (2+\cos\theta)^2$$

$$F = \varphi_\theta \cdot \varphi_t = 0$$

$$G = \varphi_t \cdot \varphi_t = 1$$

$$L = \varphi_{\theta\theta} \cdot \vec{N} = -(2+\cos t)\cos\theta \cdot \cos t\cos\theta - (2+\cos t)\sin\theta \cdot \cos t\sin\theta$$

$$= -(2+\cos t)\cos t$$

$$M = \varphi_{\theta t} \cdot \vec{N} = \sin t\sin\theta \cdot \cos t\cos\theta - \sin t\cos\theta \cdot \cos t\sin\theta = 0$$

$$N = -\cos t\cos\theta \cdot \cos t\cos\theta - \cos t\sin\theta \cdot \cos t\sin\theta - \sin t \cdot \sin t$$

$$= -(\cos^2 t\cos^2\theta + \cos^2 t\sin^2\theta + \sin^2 t) = -1$$

Hence, w.r.t $\{\varphi_\theta, \varphi_t\}$ the Gauss map has matrix

$$-\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{(2+\cos\theta)^2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (2+\cos t)\cos t & 0 \\ 0 & 1 \end{pmatrix}$$

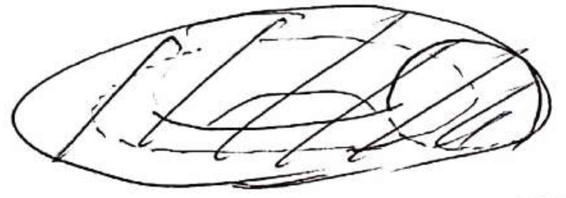
$$= \begin{pmatrix} \frac{\cos t}{2+\cos t} & 0 \\ 0 & 1 \end{pmatrix}$$

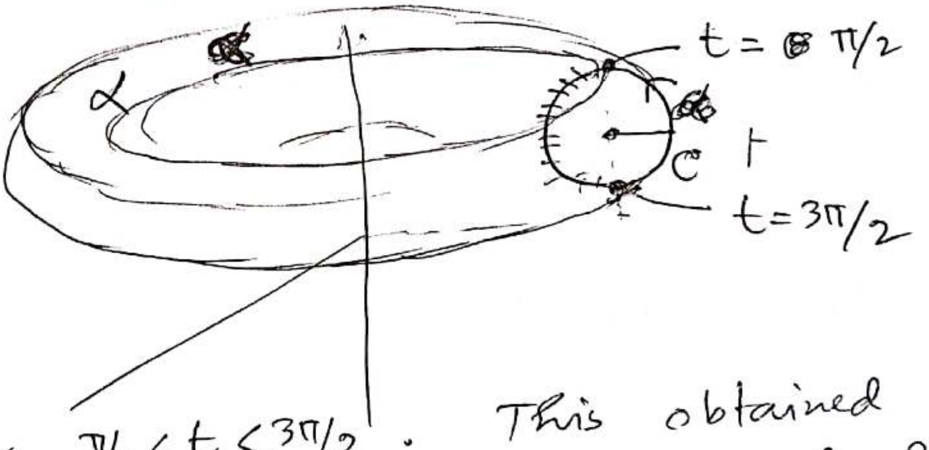
By smoothness of K, at all $\varphi(\theta, t)$,

$$\cancel{K = \frac{\cos t}{2+\cos t}} \qquad K = \frac{\cos t}{2+\cos t}.$$

$K>0$, $K=0$ or $K<0$ according as
$\cos t>0$, $\cos t=0$, $\cos t<0$ $t \in [0, 2\pi]$.

$\cos t = 0 \iff t = \pi/2, 3\pi/2$ This happens on α drawn below and its reflection in the xy-plane.





$\cos t < 0$ for $\pi/2 < t < 3\pi/2$. This obtained by revolving the shaded portion of C about the z-axis. Revolving the complementary portion of C we get all points with negative curvature.

Mean curvature $= -\frac{1}{2}\cos t\,(2+\cos t)$.

4) (i) $S_1 : x^2+y^2 = 1$

Take the line C: $y=1, x=0$ in the yz-plane with parametrization $t \mapsto (0,1,t)$

Then $\varphi(\theta,t) = (\cos\theta, \sin\theta, t)$ is a surface patch on S_1 for suitable θ-intervals.

$\varphi_\theta = (-\sin\theta, \cos\theta, 0)$

$\varphi_t = (0, 0, 1)$

$\varphi_{\theta\theta} = (-\cos\theta, -\sin\theta, 0)$

$\varphi_{\theta t} = (0,0,0)$

$\varphi_{tt} = (0,0,0)$

$$\varphi_\theta \times \varphi_t = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$$

$E = \varphi_\theta \cdot \varphi_\theta = 1$, $F = \varphi_\theta \cdot \varphi_t = 0$, $G = \varphi_t \cdot \varphi_t = 1$

$L = \varphi_{\theta\theta} \cdot \vec{N} = -1$, $M = \varphi_{\theta t} \cdot \vec{N} = 0$

$N = \varphi_{tt} \cdot \vec{N} = 0$

Hence, w.r.t $\{\varphi_\theta, \varphi_t\}$ the matrix of the Gauss map is

$$-\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence, $K = 1$, $H = -\frac{1}{2}$

4.(ii) The cone is obtained by revolving

$C: y = z \; , \; z > 0$

about the z-axis.

Choose parametrization $t \mapsto \left(0, \frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}\right)$

(unit speed) for C.

Rest is left as exercise.

5. Easy.

6. (i) Let $\varphi : \mathcal{U} \to S$ be a surface patch

and $S \xrightarrow{h} S'$ be a rigid motion.

Check : $\psi = h \circ \varphi : \mathcal{U} \to S'$ is a surface patch.

Note : • $\psi_u = Dh(\varphi_u)$

• $\psi_v = Dh(\varphi_v)$

• Dh is the same orthogonal linear map . Call it L

$$\Rightarrow \left.\begin{array}{l} \psi_u = L(\varphi_u) \\ \psi_v = L(\varphi_v) \end{array}\right\} \Rightarrow \begin{array}{l} \psi_{uu} = L(\varphi_{uu}) \\ \psi_{ut} = L(\varphi_{ut}) \\ \psi_{tt} = L(\varphi_{tt}) \end{array}$$

Note : Moreover, h being a rigid motion $\det L > 0$.

check : $\Rightarrow L(\vec{v_1} \times \vec{v_2}) = L\vec{v_1} \times L\vec{v_2} \quad \forall \vec{v_1}, \vec{v_2} \in \mathbb{R}^3$

Rest is immediate.

(ii) Left as exercise : $h(x,y,z) = c(x,y,z)$

$Dh = h$ etc.

⑦ Optional exercise.
We will remark on this later.

$C: t \mapsto (0, y(t), z(t))$

⑧ If we have a curve ~~(0,y(t),z(t))~~ in the yz- plane the surface obtained by revolving c about the z-axis has curvature

$$K = - \frac{y''}{y}$$

at any point if $y'^2 + z'^2 = 1$.

Thus to get the required surface we need to solve $-\frac{y''}{y} = -1$ i.e. $y'' = y$

and $x'^2 + y'^2 = 1$.

$y'' = y$ has solution $y(t) = ae^t + be^{-t}$ (a, b arbitrary constant).

$$\Rightarrow \quad x'^2 + (ae^t + be^{-t})^2 = 1$$

Solving this in exact form may not be possible.
Try the special case $a = 1, b = 0$:

$$x'^2 + e^{2t} = 1 \Rightarrow x' = \pm\sqrt{1 - e^{2t}}$$

Consider, $x' = \sqrt{1 - e^{2t}}$. This makes sense iff $t \le 0$.
In that case $x = \int \sqrt{1 - e^{2t}}\, dt$

(Check) $= c + \sqrt{1 - e^{2t}} - \log\left(e^{-t} + \sqrt{e^{-2t} - 1}\right)$.