Solution to HW 10

1) Suppose $f: S_1 \to S_2$ is a diffeomorphism.
Suppose $\{\varphi_i : \mathcal{U}_i \subseteq \mathbb{R}^2 \to S_1\}$ is a collection of allowable surface patches such that $\det(J(\varphi_i^{-1}\varphi_j)) > 0$ whenever $\varphi_i(\mathcal{U}_i) \cap \varphi_j(\mathcal{U}_j) \neq \phi$ on the respective domains of $\varphi_i^{-1}\varphi_j$'s.

Let $\psi_i = f \circ \varphi_i : \mathcal{U}_i \to S_2$

Claim 1: $\{\psi_i : \mathcal{U}_i \to S_2\}$ is a collection of allowable surface patches.
clearly ψ_i is smooth since φ_i, f are smooth for each i. $f: \varphi_i(\mathcal{U}_i) \to f(\varphi_i(\mathcal{U}_i))$ is a homeomorphism since f is a diffeomorphism.
Thus $\psi_i : \mathcal{U}_i \to \psi_i(\mathcal{U})$ is a homeomorphism.

Now, $\psi_{iu} = Df(\varphi_{iu})$

$\psi_{iv} = Df(\varphi_{iv})$

We know that diffeomorphism induces linear isomorphism between tangent spaces. Since $\varphi_{iu}, \varphi_{iv}$ are linearly independent so are ψ_{iu}, ψ_{iv}. Hence, $\psi_{iu} \times \psi_{iv} \neq \vec{0}$.

Claim 2: $\det(J(\psi_i^{-1}\psi_j)) > 0$. This is clear:

Note that $\psi_i^{-1} \circ \psi_j = (f \circ \varphi_i)^{-1} \circ (f \circ \varphi_j) = \varphi_i^{-1}\varphi_j$.

② This is left for interested people. ②

3) We know that the surface has patches of the form

$$\varphi_{a,b}: (a, \cancel{\ }b) \times I \longrightarrow S$$
$$(\theta, t) \longmapsto (t\cos\theta, t\sin\theta, g(t))$$

where $I =$ domain of g, & $b-a \leq 2\pi$.

~~Then~~ Let $\mathcal{U}_{a,b} = (a,b)\times I$.

Now, two such surface patches intersect iff $(a,b) \cap (c,d) \neq \phi$.

Check that the transition maps are identity.

4) S is the level surface $f(x,y,z) = d$ where $f = ax+by+cz$. Now, $\text{grad}(f) = (a,b,c)$ etc.

5) ~~Hence~~ Ⓐ $S: \quad x^2+y^2-z=0$

Let $f = x^2+y^2-z$. $\quad \text{grad}(f) = (2x, 2y, -1)$

$\Rightarrow \text{grad}(f)_{(1,1,2)} = (2,2,-1)$

Thus $T_pS: \quad 2x+2y-z=0$

Let $\vec{u} = (1,0,2)$, $\vec{v} = (0,1,2)$. Then ~~TpS =~~ $\{\vec{u}, \vec{v}\}$ is a basis of T_pS.

Now, f is the restriction of the smooth map $F: \mathbb{R}^3 \to \mathbb{R}^2$ which has
$(x,y,z) \mapsto (x,y)$

Jacobian $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ at all points.

Hence, if α is any curve in S passing through $p = (1,1,2)$, $\alpha'(p) = \vec{u}$ then

$$Df_p(\vec{u}) = DF_p(\vec{u}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Similarly, $Df_p(\vec{v}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(B) Very similar to (A).

(C) In this case let $\mathcal{U} \subseteq \mathbb{R}^3$ be the open set $\{(x,y,z) ; z > 0\}$. Then $S \subseteq \mathcal{U}$ and f is the restriction of

$$F: \mathcal{U} \to \mathbb{R}^2$$
$$(x,y,z) \mapsto \left(\frac{x}{z}, \frac{y}{z}, z\right)$$

Now, do as in (A).

6. (i) Df_p ~~dep~~ depends on the values of f ~~on~~ "near" p, i.e. ~~on~~ values of f on any open set $\mathcal{U} \ni p$.

If $f : \mathcal{U} \to f(\mathcal{U})$ is a diffeomorphism then $g := f^{-1} : f(\mathcal{U}) \to \mathcal{U}$ is smooth.

Now, $g \circ f : \mathcal{U} \to \mathcal{U}$ is the identity.
Using chain rule

$$Dg_{f(p)} \circ Df_p = Id \; : T_pS_1 \to T_pS_1$$

* $\boxed{\text{Note: } T_p\mathcal{U} = T_pS_1}$

~~Thus Df_p is injective. Since~~

Similarly $Df_p \circ Dg_{f(p)} : T_{f(p)}S_2 \to T_{f(p)}S_2$ is identity. Thus $Dg_{f(p)} = Df_p^{-1}$.

(ii) Let $\varphi : \mathcal{U} \to S_1$ be a surface patch, $p \in \varphi(\mathcal{U})$ such that there is a surface patch $\psi : V \to S_2$ where $f \circ \varphi(\mathcal{U}) \subseteq \psi(V)$. Let $\varphi^{-1}(p) = x$

Let $F = \psi^{-1} \circ f \circ \varphi : \mathcal{U} \to V$.
Since φ, ψ are diffeomorphisms $DF_x : T_x\mathcal{U} \to T_{F(x)}V$ is an isomorphism or $\det(JF_x) \neq 0$.

~~Now, f being smooth, the map $h : \mathcal{U} \to \mathbb{R}$ defined by $(u,v) \mapsto \det JF_{(u,v)}$ is a smooth function. Since $h(x) \neq 0$ there is an op~~

By inverse function theorem there is an open set $\mathcal{U}_1 \subseteq \mathcal{U}$ such that $F : \mathcal{U}_1 \to F(\mathcal{U}_1)$ is a diffeomorphism, $p \in \mathcal{U}_1$, $F(\mathcal{U}_1) \subseteq V$ open.
check: $f : \varphi(\mathcal{U}_1) \to f \circ \varphi(\mathcal{U}_1)$ is a diffeomorphism.

(7) Do it yourself.