Explanation for the signed curvature formula

Some of you were asking me about the following formula for the signed curvature, which you have come across on the internet.

$$\frac{\dot{\gamma}(t) \times \ddot{\gamma}(t)}{\|\dot{\gamma}(t)\|^3}$$

But the signed curvature is a scalar not a vector as this formula seems to imply, so one needs to know how to interpret it. Furthermore, the above vector is orthogonal, and not in the plane in which the curve lies (the signed curvature makes sense *only* if the curve is planar), let alone be in the direction of the signed normal. Also, the cross product seems flipped when compared with a similar expression in our formula for curvature. All of these are explained below.

Exercises 2. and 3. a) from problem set 3 show that, given a prametrization γ, you can compute the acceleration *after you reparamtrize* to a unit speed parametrization $\tilde{\gamma}$ by,

$$\ddot{\tilde{\gamma}}(s(t)) = \frac{\dot{\gamma}(t) \times (\ddot{\gamma}(t) \times \dot{\gamma}(t))}{\|\dot{\gamma}(t)\|^4}$$

Do not forget that a point $\gamma(t)$ is represented by $\tilde{\gamma}(s(t))$ when using the arc length paramatrization. In other words, the paramater t should be changed to $s(t)$ to refer to the same point if you are using the arc length parametrization $\tilde{\gamma}$ instead of γ. Until this point, the derivation of the curvature and signed curvature formulae are the same.

If you all you are interested in is the norm, then part b) of exercise 3 allows you to simplify this formula very easily to obtain the curvature. However, the signed curvature needs more work to derive as well as to interpret!

The above formula for $\ddot{\tilde{\gamma}}(s(t))$ must be a vector normal to the unit tangent vector $\mathbf{T}(s(t)) = \dot{\tilde{\gamma}}(s(t)) = \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}$ (again, note the $s(t)$ since we can only speak of the unit tangent and signed normal etc. when using an arc length parametrization).

Using the above to replace the first $\dot{\gamma}(t)$ by $\|\dot{\gamma}(t)\|\mathbf{T}(s(t))$, we can now write the acceleration vector as:

$$\ddot{\tilde{\gamma}}(s(t)) = \frac{\mathbf{T}(s(t)) \times (\ddot{\gamma}(t) \times \dot{\gamma}(t))}{\|\dot{\gamma}(t)\|^3}$$

or equivalently, and what will prove more useful to compare it with the formula that you have seen, as

$$\ddot{\tilde{\gamma}}(s(t)) = \mathbf{T}(s(t)) \times \frac{\ddot{\gamma}(t) \times \dot{\gamma}(t)}{\|\dot{\gamma}(t)\|^3}$$

Observe that $\frac{\ddot{\gamma}(t)\times\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|^3}$ is normal to the plane so let us write it as a scalar multiple of the signed unit normal to the plane, i.e. $c(t)\hat{k}$, where $\hat{k} := \mathbf{T}(s(t)) \times \mathbf{N}_s(s(t))$ (this is constant because the curve lies in a fixed plane!). The formula is now:

$$\ddot{\tilde{\gamma}}(s(t)) = \mathbf{T}(s(t)) \times c(t)\hat{k} = c(t)(\mathbf{T}(s(t)) \times \hat{k})$$

Now, try to use the right hand screw rule to see why $\mathbf{T}(s(t)) \times \hat{k} = -\mathbf{N}_s(s(t))$ (You have used a similar reasoning when deducing all the cross product relations between $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ for space curves). Note the minus sign! Therefore,

$$\ddot{\tilde{\gamma}}(s(t)) = -c(t)\mathbf{N}_s(s(t))$$

This proves that $c(t) = -\kappa_s(t)$. Since we can, why not remove the negative sign (which we get because $\kappa_s(t) = -c(t)$), by just flipping the cross product, to get $\frac{\dot{\gamma}(t)\times\ddot{\gamma}(t)}{\|\dot{\gamma}(t)\|^3}$. Note that $c(t)$ was simply the coefficient when writing $\frac{\ddot{\gamma}(t)\times\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|^3}$ in terms of $\hat{k}$, which it is parallel to.

So this is how you interpret this formula, even though it is a vector: if your calculations are correct, *the cross product* $\frac{\ddot{\gamma}(t)\times\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|^3}$ *will be a coefficient of* $\hat{k}$*. That coefficient is the signed curvature* as has been demonstrated above.