MTH 201, Curves and surfaces

Practice problem set 9

1. Consider a ruled surface defined as follows: γ is a unit speed parametrization of the unit circle $x^2 + y^2 - 1 = 0$. Let $\delta(t) = \dot{\gamma}(t) + \mathbf{w}$ where $\mathbf{w}$ is the unit vector $(0, 0, 1)$.

   a) Prove that $\sigma(t, x) = \gamma(t) + x\delta(t)$, is a surface patch of the surface $x^2 + y^2 - z^2 = 1$, thereby showing that it is a ruled surface.
   b) Let $\delta'(t) = -\dot{\gamma}(t) + \mathbf{w}$. Prove that $\sigma(t, x) = \gamma(t) + x\delta'(t)$, is a surface patch of the same surface.

2. Prove that the surface patch $\sigma(t, x) = (1 + x)\gamma(t) - x\mathbf{v}$ of a generalized cone with vertex $\mathbf{v}$ and over a curve parametrized by γ, is injective if and only if any straight line containing the vertex can intersect the curve in at most one point.

3. Find two surface patches to cover the mobius band. Compute the transition functions and the determinant of their Jacobians. Show that one of them is positive and the other is negative.

4. Prove that the mobius band is non-orientable.

5. Prove that any smooth surface which contains an open set diffeomorphic to the mobius band, is non-orientable.

6. Prove that if $f : S_1 \to S_2$ is a diffeomorphism, then S_1 is orientable if and only if S_2 is orientable. What if f was only a local diffeomorphism?

7. Prove that if a function $\sigma : U \to \mathbb{R}^3$ satisfies the condition that $\sigma_x \times \sigma_y \neq 0$ at a point $p \in U$, then there exists an open set U_1, also containing p, so that σ restricted to U_1, which is a subset of U, is injective. *(Hint: Inverse function theorem. During the lectures we saw two ways to get around the problem that the given function has a domain and range of different dimensions. Both of them should work!)*