# MTH 201, Curves and surfaces

## Practice problem set 8

1. Show that the set $S = \{(x, y, z) \mid 2x^2 + y^2 - z^2 - 1\}$ is a surface. Do not find an explicit surface patch but use the theorem for level sets (i.e. defined by $\{(x, y, z) \mid F(x, y, z) = 0\}$)

2. $\sigma(x, y) = ((1 + x\sin(y/2))\cos(y), (1 + x\sin(y/2))\sin(y), x\cos(y/2))$ be a surface patch defined when $-1/2 < x < 1/2$ and $-\pi < y < \pi$.

   a) Compute $\mathbf{n}(x, y) := \sigma_x(x, y) \times \sigma_y(x, y)$.
   b) Show that $\lim\limits_{y \to \pi} \mathbf{n}(0, y) = -\lim\limits_{y \to -\pi} \mathbf{n}(0, y)$

3. Let $\sigma(x, y) = (f(x)\cos(y), f(x)\sin(y), g(x))$. Compute $\sigma_x \times \sigma_y$. When is it zero?

4. If $\sigma$ is a surface patch and $\gamma(t) = \sigma(x(t), y(t))$, where $x : \mathbb{R} \to \mathbb{R}$ and $y : \mathbb{R} \to \mathbb{R}$ are smooth functions, then prove that $\dot{\gamma} = \dot{x}\sigma_x + \dot{y}\sigma_y$.

5. If $\sigma$ is a surface patch and $\gamma(t) = \sigma(a + \alpha t, b + \beta t)$, then prove that $\dot{\gamma} = \alpha\sigma_x + \beta\sigma_y$ at $t = 0$.

6. Recall the definition of a smooth function $f : S \to \mathbb{R}$. Prove that if $\pi(x, y, z) = x$ is restricted to the surface $S$, then it is a smooth function from $S$ to $\mathbb{R}$.

7. Recall the definition of $D_p(f) : T_p(S) \to T_p(S)$, where $T_p(S)$ denotes the tangent space at $p$ of a surface $S$, and prove the following:

   a) $D_p(\mathrm{id_S}) = \mathrm{id}_{\mathrm{T_p(S)}}$. ($id_A : A \to A$ is the identity map on $A$).
   b) If $f : S_1 \to S_2$ and $g : S_2 \to S_3$ are smooth maps then $D_p(g \circ f) = D_{f(p)}(g) \circ D_p(f)$.
   c) If $f$ is a diffeomorphism, then $D_p(f)$ is invertible.

8. Consider the set $S := \{(x, y, z) \mid F(x, y, z) = 0\}$. Define $f : \mathbb{R}^3 \to \mathbb{R}^3$ by $f(x, y, z) = (x, y, F(x, y, z))$

   a) Prove that if $F_z \neq 0$ for some $p = (x_0, y_0, z_0)$ in $S$, then there exists some open neighbourhood $U$ around $p$ and an open neighbourhood $V$ around $f(p)$ so that $f : U \to V$ has a smooth inverse $g : V \to U$.
   b) Prove that the image of $V \cap \{(x, y, z) \mid z = 0\}$ under $g$ is in $S$. Use this observation to define a regular surface patch of $S$ around $p$. To check regularity you will need to examine the coordinates of $g$.

c) Can you find a condition to impose on $F$ so that $S$ is a smooth surface?

9. Let $\sigma : U \to S \cap W$ (here, $W$ is open in $\mathbb{R}^3$) be a regular surface patch around a point $p = \sigma(x_0, y_0)$ and define $\pi_1(x, y, z) = (y, z)$, $\pi_2(x, y, z) = (x, z)$, $\pi_3(x, y, z) = (x, y)$.

   a) Prove that for some $i = 1, 2, 3$, $\pi_i$ is injective when restricted to $\sigma(U')$, where $U'$ is an open subset (possibly different from $U$) containing $(x_0, y_0)$. In fact, prove this as a consequence of the fact that $\pi_i \circ \sigma$ is a diffeomorphism when restricted to $U'$. Will the same $\pi_i$ work for all the points? Prove it or find a counterexample.
   b) Let $\gamma : (a, b) \to W \cap S \subset \mathbb{R}^3$ parametrize a curve lying on the intersection of the surface $S$ with an open subset $W$. Let $\tilde{\gamma} : (a, b) \to U \subset \mathbb{R}^2$ denote the curve in $U$ such that $\sigma \circ \tilde{\gamma} = \gamma$. Use the previous part to show that $\tilde{\gamma}$ is smooth.
   c) Let $\sigma' : U' \to W \cap S$ denote another surface patch. Why is $\sigma^{-1} \circ \sigma' = (\pi_i \circ \sigma)^{-1} \circ (\pi_i \circ \sigma')$? Use that to prove that $\sigma^{-1} \circ \sigma'$ is smooth. Why could it not be done directly?