MTH 201, Curves and surfaces

Practice problem set 10

1. Compute the first fundamental form with respect to the surface patch $\sigma(x, y) = (f(x)\cos(y), f(x)\sin(y), g(x))$. Why do you think your answer does not involve g?

2. Find a surface patch for each of the following surfaces, and compute the first fundamental form of that patch.

   a) A sphere. $\sigma(x, y) = (x, y, \sqrt{1 - x^2 - z^2})$
   b) A generalized cylinder over a plane curve γ. $\sigma(x, y) = \gamma(x) + y\delta(x)$, where δ is orthogonal to the plane that contains γ.

3. If σ is a regular surface patch of a surface, $\tilde{\sigma} = \sigma \circ \Theta$, another surface patch, where Θ is a coordinate transformation, then show that their respective first fundamental forms $E\mathrm{d}x^2 + 2F\mathrm{d}x\mathrm{d}y + G\mathrm{d}y^2$ and $\tilde{E}\mathrm{d}\tilde{x}^2 + 2\tilde{F}\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} + \tilde{G}\mathrm{d}\tilde{y}^2$ are related by

$$\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = (\mathrm{Jac}(\Theta))^t \begin{pmatrix} E & F \\ F & G \end{pmatrix} \mathrm{Jac}(\Theta)$$

4. Let γ be the unit speed parametrization of a regular *space* curve.

   a) Prove that the surface patch $\sigma(x, y) = \gamma(x) + y\dot{\gamma}(x)$ is regular, then the curvature κ is strictly positive.
   b) Prove that if γ is a *plane* curve, then σ is a surface patch of a plane.
   c) Compute the first fundamental form of σ (γ may be non-planar). Show that there exists a *plane* curve $\tilde{\gamma}$ so that $\tilde{\sigma}(x, y) = \tilde{\gamma}(x) + y\dot{\tilde{\gamma}}(x)$ has the same first fundamental form as σ.

5. Prove that a local diffeomorphism, $f : S_1 \to S_2$, is a local isometry if and only given *any* surface patch σ_1 of S_1, its fundamental form is equal to the fundamental form of the surface patch $f \circ \sigma$ of S_2 by proving the following parts:

   a) $(f \circ \sigma)_x = D_p(f)(\sigma_x)$. (Remember that σ_x can be interpreted as the derivative of some curve on the surface. That should help you to link it up with the definition of $D_p(f)$.
   b) Use part a) to prove that $\langle (f \circ \sigma)_x, (f \circ \sigma)_x \rangle = f^* \langle \sigma_x, \sigma_x \rangle$
   c) Using the fact that f is a local isometry if and only if $\langle D_p(f)(\mathbf{v}), D_p(f)(\mathbf{w}) \rangle_p = \langle \mathbf{v}, \mathbf{w} \rangle_{f(p)}$, and using the previous parts, prove that the first fundamental forms are the same.