# Solution to HW 9



1. (i) $S$ is given by $f(x,y,z) = -z + x^2 + y^2 = 0$

and $\text{grad}(f) = (2x, 2y, -1) \neq 0$ on $S$.

Hence, $\text{grad}(f)(p) = (2, 0, -1)$ is normal to $S$ at $p$.

Hence,

$$T_p S = \{(x,y,z) \mid (x,y,z)\cdot(2,0,-1) = 0\}$$

$$= \text{plane } 2x - z = 0$$

The tangent plane is given by

$$\left((x,y,z) - \overrightarrow{Op}\right) \cdot (2,0,-1) = 0$$

i.e. $2(x-1) - (z-1) = 0$ or $2x - z - 1 = 0$.

(ii) $S$ is given by $F(x,y,z) = f(x) - y = 0$

and $\text{grad}\, F = (f'(x), -1, 0) \neq 0$ on $S$.

If $p = (x_0, y_0, z_0) \in S$ then $\text{grad}\, F(p) = (f'(x_0), -1, 0)$

Hence, $T_p S =$ the plane $f'(x_0)x - y = 0$

and the tangent plane : $f'(x_0)(x - x_0) - (y - y_0) = 0$

(iii) Any point of $S$ is of the form $(y\cos\theta, y\sin\theta, f(y))$. ~~where~~

$\varphi(\theta, y) = (y\cos\theta, y\sin\theta, f(y))$ gives an allowable surface patch when $\theta$ ~~is~~ varies over an ~~~~ open interval of length $\leq 2\pi$.

②

$$\varphi_y = (\cos\theta, \sin\theta, f'(y))$$

$$\varphi_\theta = (-y\sin\theta, y\cos\theta, 0)$$

$$\Rightarrow \quad \varphi_y \times \varphi_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & f'(y) \\ -y\sin\theta & y\cos\theta & 0 \end{vmatrix}$$

$$= -yf'(y)\cos\theta\,\vec{i} - yf'(y)\sin\theta\,\vec{j} + y\,\vec{k}$$

$$= -yf'(y)(\cos\theta\,\vec{i} + \sin\theta\,\vec{j}) + y\,\vec{k}$$

$y \neq 0$ since the curve does not touch z-axis.

Hence, at ~~a~~ point $(y_0\cos\theta_0, ~~~~ y_0\sin\theta_0, f(y_0))$

$$~~~~ f'(y_0)(\cos\theta_0\,\vec{i} + \sin\theta_0\,\vec{j}) + ~~~~ \vec{k}$$

is normal to the surface.

Hence, ~~~~ if $p = (y_0\cos\theta_0, y_0\sin\theta_0, f(y_0))$

then

$T_pS$ = the plane ~~~~ $f'(y_0)(\cos\theta_0\, x + ~~~~ \sin\theta_0\, y)$
$+ z = 0$

and the tangent plane to $S$ at $p$ is the plane

$$f'(y_0)\cos\theta_0(x - y_0\cos\theta_0) + f'(y_0)\sin\theta_0(y - y_0\sin\theta_0) + z - f(y_0) = 0$$

$$\Rightarrow \quad f'(y_0)(\cos\theta_0\, x + \sin\theta_0\, y) + z - f(y_0) - f'(y_0) = 0.$$

2. (i), (iii): The functions are restrictions of smooth functions defined on the whole of $\mathbb{R}^3$. Hence they are smooth.

(iv) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

Hence $f$ is given by $f(x,y,z) = (x^2-y^2, 2xy, z)$ which is smoothly extendable to $\mathbb{R}^3$.

Hence $f$ is smooth.

(ii) For all points $(x,y,z)$, $x \neq 0$, $y \neq 0$ $f$ is smooth. We claim that $f$ is not smooth when $x=y=0$ which corresponds to the ~~po~~ poles $(0,0,\pm 1)$.

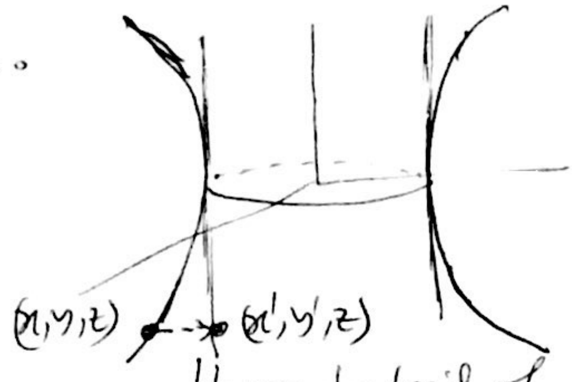
Let ~~us~~ ~~the~~ check that at $p = (0,0,1)$ $f$ is not smooth.

Suppose, if possible, that $f$ is smooth at $p$. We know that $\varphi: \mathbb{D}^2 \to S^2$, $(x,y) \mapsto (x, y, \sqrt{1-x^2-y^2})$ is a surface path, allowable too. ~~Th~~

Note: $\varphi(0,0) = p$. Hence, $f \circ \varphi$ should be smooth at $(0,0)$. But $f \circ \varphi(x,y) = \sqrt{x^2+y^2}$ is not smooth at $(0,0)$.

<u>Note</u>: $\mathbb{D}^2 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$.

3.




Hyperboloid of one sheet.

Clearly the cylinder is "inside" the hyperboloid. We define a map from the hyperboloid say $S_1$ to the cylinder say $S_2$ by mapping $(x, y, z)$ to a point of the form $(x', y', z)$ (See figure).

[Note: Any point of the hyperboloid $S_1$ is of the form

$$\left(\sqrt{1+z^2}\cos\theta,\ \sqrt{1+z^2}\sin\theta,\ z\right)$$

because $S_1$ is obtained by revolving $y^2 - z^2 = 1$, $y \geq 0$, about the $z$-axis.

$y^2 - z^2 = 1$, $y \geq 0$ is equivalent to $y = \sqrt{1+z^2}$.]

Clearly the map $f : S_1 \longrightarrow S_2$ is given by

$$f(x, y, z) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, z\right)$$

This is smooth on $S_1$ because $x^2 + y^2 \neq 0$ on $S_1$ and $f$ is the restriction of the smooth map

$$\mathbb{R}^3 \setminus \{z\text{-axis}\} \longrightarrow \mathbb{R} \quad (x, y, z) \longmapsto \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, z\right)$$

Ex: check that $g : S_2 \to S_1$

$$(x, y, z) \longmapsto \left(\sqrt{1+z^2}\, x,\ \sqrt{1+z^2}\, y,\ z\right)$$

is the inverse of $f$ and $g$ is smooth too.