

1. Can you think of two curves  $\alpha, \beta: \mathbb{R} \rightarrow \mathbb{R}^2 \subseteq \mathbb{R}^3$  such that  $\alpha$  cannot be taken to  $\beta$  by any rigid motion of  $\mathbb{R}^3$  although  $\alpha, \beta$  have the same curvature function  $\kappa: \mathbb{R} \rightarrow \mathbb{R}$ ?  
Give an intuitive idea.

2. Let  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined as follows:

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Show that  $f$  is discontinuous at  $(0, 0)$  but  $f$  has directional derivatives ~~in~~ along any vector  $v = (a, b) \in \mathbb{R}^2$  at  $p = (0, 0)$ .

3. Show that  $f$  is  $C^1 \Rightarrow f$  is  $C^0$  where  $f: U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $U$  open in  $\mathbb{R}^n$ .

(See a reference if needed.)

4. Find directional derivatives of the functions  $f$  given below at the point along the vector  $v$ .

(i)  $f(x, y, z) = x^2 + yz + xz^2$ ,  $p = (1, 1, 1)$ ,  $v = (1, 2, 3)$ .

(ii)  $f(x, y, z) = xy + yz + zx$ ,  $p = (1, 0, 1)$ ,  $v = (0, 1, 1)$ .

(iii)  $f(x, y) = xy(x+y)$ ,  $p = (1, 2)$ ,  $v = (1, 1)$ .

5. For a smooth function  $f: U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$

show that  $f'(p; v) = \text{grad}(f)(p) \cdot v$

$\forall p \in U$  and  $v \in \mathbb{R}^n$ .

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6. For a smooth function  $f: U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  
 $p \in U$  and a smooth curve  $\alpha: (-\epsilon, \epsilon) \rightarrow U$   
with  $\alpha(0) = p$  show that

$$(f \circ \alpha)'(0) = Jf(\alpha(0)) \cdot \alpha'(0).$$

using the chain rule.

7. Check if the following are diffeomorphisms  
onto the image

(i)  $f: \mathbb{R} \rightarrow \mathbb{R} \quad x \mapsto e^x$

(ii)  $f: \mathbb{R} \rightarrow \mathbb{R} \quad x \mapsto \frac{x^2}{x^2+1}$

(iii)  $f: \mathbb{R} \rightarrow \mathbb{R} \quad x \mapsto x^3$

(iv)  $f: \{(x, y) \in \mathbb{R}^2 : x > 0, -\pi < y < \pi\} \rightarrow \mathbb{R}^2 \quad (x, y) \mapsto (x \cos y, x \sin y)$

8. Check if the following are allowable surface  
patches: First check if they are surface patches.

(i)  $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \quad (x, y) \mapsto (x, y, xy)$

(ii)  $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \quad (x, y) \mapsto (x, y^2, y^3)$

(iii)  $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \quad (x, y) \mapsto (x+x^2, y, y^4)$

(iv)  $\varphi: \mathbb{R}^2 \setminus \{(0,0)\} \rightarrow \mathbb{R}^3 \quad (x, y) \mapsto (x, y, \sqrt{x^2+y^2})$

(v)  $\varphi: (0, 2\pi) \times \mathbb{R} \rightarrow \mathbb{R}^3 \quad (x, y) \mapsto (\cos x, \sin x, y)$

(vi)  $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \quad (x, y) \mapsto (x^2, xy, y^2)$