Solution to HW6

1) Suppose $\alpha : I \to \mathbb{R}^2$ is such a curve. Then κ_s is also constant. Therefore, if φ is the turning angle we have

$$\frac{d\varphi}{ds} = \text{constant.} = c \text{ say.}$$

Case 1 : $c = 0$

Then φ is constant

$\Rightarrow \vec{T} = (\cos\varphi(s), \sin\varphi(s))$ is a constant vector.

If $\vec{T} = (a_1, a_2)$ then

$$\alpha(s) = (a_1 s + b_1, a_2 s + b_2) \text{ for some}$$

constants b_1, b_2.

Note: $(a_1, a_2) \neq (0,0)$ ~~else since other~~ since $a_1^2 + a_2^2 = 1$.

Clearly α is contained in the plane

$$a_2(x - b_1) = a_1(y - b_2).$$

Case 2 : $c \neq 0$

$$\varphi = cs + c_0 \quad \text{for some constant } c_0$$

$$\Rightarrow \vec{T}(s) = (\cos(cs + c_0), \sin(cs + c_0))$$

$$\Rightarrow \alpha(s) = \left(\frac{1}{c}\sin(cs + c_0), -\frac{1}{c}\cos(cs + c_0)\right) + (c_1, c_2)$$

for some constants c_1, c_2

Clearly α is contained in the circle

$$(x - c_1)^2 + (y - c_2)^2 = 1/c^2$$

(2)

2) $\kappa_s = \cos s \Rightarrow \frac{d\varphi}{ds} = \cos s$

$\Rightarrow \varphi(s) = \sin s + c_0$ for some constant c_0

$\Rightarrow \vec{T}(s) = \left(\cos(\sin s + c_0), \sin(\sin s + c_0)\right)$

$P = (0,0)$ is on the curve. If length is measured from P then

$$\vec{T}(0) = (\cos c_0, \sin c_0) = (1,0)$$

Hence, we can take $c_0 = 0$.

$\Rightarrow \vec{T}(s) = \left(\cos(\sin s), \sin(\sin s)\right)$

$\Rightarrow \alpha(s) = \left(\int_0^s \cos(\sin u)\,du, \int_0^s \sin(\sin u)\,du \right)$

3) (i) $\Rightarrow$ (ii): Let $e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$.

(*) $Ae_i = i^{th}$ column vector of A.

From $AA^t = I$ deduce that $\{Ae_1, Ae_2, Ae_3\}$ is an orthonormal basis of $\mathbb{R}^3$.

Given $v, w \in \mathbb{R}^3$ write $v = a_1e_1 + a_2e_2 + a_3e_3$, $w = b_1e_1 + b_2e_2 + b_3e_3$, $a_i, b_i \in \mathbb{R}$ and then use that $v \mapsto Av$ is a linear map.

(ii) $\Rightarrow$ (i): Apply (ii) for $v, w \in \{e_1, e_2, e_3\}$. Then use (*).

(ii) $\Rightarrow$ (iii): Immediate

(iii) $\Rightarrow$ (ii): Compute $A(v+w) \cdot A(v+w)$

4. (i) Check it for $v, w \in \{e_1, e_2, e_3\}$ or $\{\vec{i}, \vec{j}, \vec{k}\}$ and then use linearity.

(ii) Do it yourself.

5. (i) $\alpha(t) = (t, t^2, t^3)$

$$\Rightarrow \left.\begin{aligned} \alpha'(t) &= (1, 2t, 3t^2) \\ \alpha''(t) &= (0, 2, 6t) \\ \alpha'''(t) &= (0, 0, 6) \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha'(0) &= (1,0,0) = \vec{i} \\ \alpha''(0) &= (0,2,0) = 2\vec{j} \\ \alpha'''(0) &= (0,0,6) = 6\vec{k} \end{aligned}$$

clearly $\vec{T}(0) = \vec{i}$

$\vec{b}$ is parallel to $\alpha'(t) \times \alpha''(t)$.

$$\Rightarrow \vec{b}(0) = \frac{\vec{i} \times 2\vec{j}}{\| \vec{i} \times 2\vec{j} \|} = \vec{k}$$

$$\Rightarrow \vec{n}(0) = \vec{b}(0) \times \vec{T}(0) = \vec{k} \times \vec{i} = \vec{j}.$$

$$\kappa(0) = \frac{\| \alpha'(0) \times \alpha''(0) \|}{\| \alpha'(0) \|^3} = 2$$

$$\tau(0) = \frac{(\alpha'(0) \times \alpha''(0)) \cdot \alpha'''(0)}{\| \alpha'(0) \times \alpha''(0) \|^2} = \frac{2\vec{k} \cdot 6\vec{k}}{\| 2\vec{k} \|^2} = 3$$

(ii) $\alpha(t) = (t\cos t, t\sin t, t)$

$$\Rightarrow \alpha'(t) = (\cos t - t\sin t, \sin t + t\cos t, 1),$$

$$\alpha''(t) = (-2\sin t - t\cos t, 2\cos t - t\sin t, 0)$$

$$\alpha'''(t) = (-3\cos t + t\sin t, \cancel{\cos t} - 3\sin t - t\cos t, 0)$$

$$\Rightarrow \alpha'(0) = (1, 0, 1) = \vec{i} + \vec{k}$$

$$\left.\begin{aligned} \alpha''(0) &= (0, 2, 0) \\ &= 2\vec{j} \end{aligned}\right\}, \quad \alpha'''(0) = (-3, 0, 0) = -3\vec{i}$$

etc.

6. (i) Yes, This curve is on the plane $z = x + y + 1$.

(ii) NO. Calculate torsion and show that it is not zero. OR use that e is a transcendental number