

# Solution to HWS

①

2) If  $\vec{T} = (\cos \varphi(s), \sin \varphi(s))$  then  $\kappa_s = \frac{d\varphi}{ds}$  and

(i)  $\vec{n}_s = (-\sin \varphi(s), \cos \varphi(s))$ .

$$\Rightarrow \frac{d\vec{n}_s}{ds} = -\varphi'(s) (\cos \varphi(s), \sin \varphi(s)) = -\kappa_s \vec{T}.$$

(ii) Since  $\varphi$  is smooth,  $\kappa_s$  is smooth.

3) Signed Curvature of  $\beta = \frac{\vec{r} \cdot (\beta'(t) \times \beta''(t))}{\|\beta'(t)\|^3}$

$$= \frac{\vec{r} \cdot (-\alpha'(a+b-t) \times \alpha''(a+b-t))}{\|\alpha'(a+b-t)\|^3} = -\text{Signed}$$

curvature of  $\alpha$  at  $(a+b-t)$

Note:  $\beta(a) = \alpha(b)$ ,  $\beta(b) = \alpha(a)$   $\beta$  is the "same" curve traversed in the opposite direction.

Roughly speaking signed curvature depends not only on the "shape" of the curve but also on the direction.

4)  $x = a \cos t$ ,  $y = b \sin t$  is a parametrization of the ellipse.

Here,  $\alpha(t) = (a \cos t, b \sin t)$

$$\Rightarrow \alpha'(t) = (-a \sin t, b \cos t),$$

$$\alpha''(t) = (-a \cos t, -b \sin t)$$

$$\Rightarrow \alpha' \times \alpha'' = ab \vec{z}$$

$$\Rightarrow \kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3} = \frac{ab}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}}$$

$$= \frac{ab}{\sqrt{(a^2 - b^2) \sin^2 t + b^2}}$$

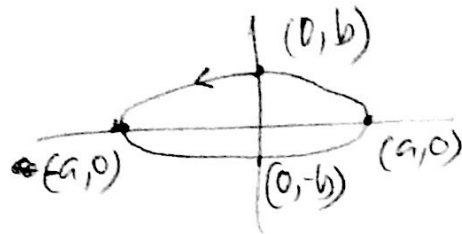
Hence  $\kappa$  is maximum for  $\sin t = 0$  and minimum for  $\sin t = \pm 1$

In  $[0, 2\pi]$ ,  $\sin t = 0 \Leftrightarrow t = 0, \pi, 2\pi$

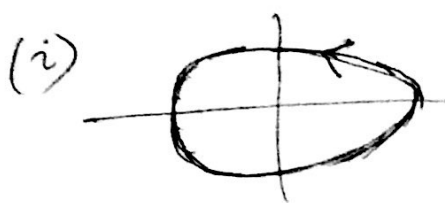
(2)

and  $\sin t = \pm 1 \Leftrightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}$

It follows that curvature is maximum at the points  $(\pm a, 0)$  and minimum at  $(0, \pm b)$



5)



$$\kappa = \frac{|r^2 + 2r'^2 - rr''|}{(r^2 + r'^2)^{3/2}}$$

$$r = 2 + \cos \theta \Rightarrow r' = -\sin \theta, r'' = -\cos \theta$$

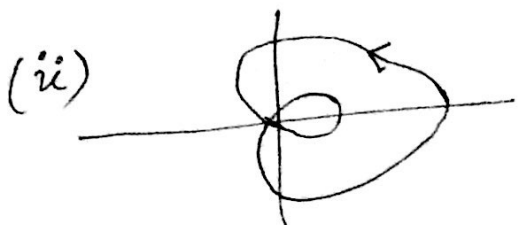
$$\Rightarrow r^2 + r'^2 = (2 + \cos \theta)^2 + \sin^2 \theta = 4 + 3 \cos \theta$$

$$r^2 + 2r'^2 - rr'' = (2 + \cos \theta)^2 + 2 \sin^2 \theta + \cos \theta (2 + \cos \theta)$$

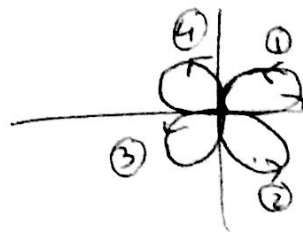
$$= 4 + 4 \cos \theta + \cos^2 \theta + 2 \sin^2 \theta + 2 \cos \theta + \cos^2 \theta$$

$$= 6 + 8 \cos \theta$$

Hence  $\kappa = \frac{|6 + 8 \cos \theta|}{(4 + 3 \cos \theta)^{3/2}}$



(iii)



The arrows show the direction as  $\theta$  increases from 0 to  $2\pi$ .

6) Apply  $\kappa = \frac{|y'''|}{(1 + y'^2)^{3/2}}$

~~...~~  
~~...~~

~~A [ ... ]~~  $\Rightarrow$  A [ ... ]