Solution to HW-11

1. Fix a surface patch $\varphi : \mathcal{U} \to S$ around p.
~~If~~ Then w.r.t. $\{\varphi_u, \varphi_v\}$ the Gauss map T has matrix

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = -\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

The characteristic ~~equation~~ polynomial of $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is

$$\det\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix} - \lambda I\right)$$

$$= \det\left(\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix} + \lambda I\right)$$

$$= \frac{1}{\det\begin{pmatrix} E & F \\ F & G \end{pmatrix}} \det\left(\begin{pmatrix} L & M \\ M & N \end{pmatrix} + \lambda\begin{pmatrix} E & F \\ F & G \end{pmatrix}\right)$$

$$= \frac{1}{\det\begin{pmatrix} E & F \\ F & G \end{pmatrix}} \det\begin{pmatrix} L+\lambda E & M+\lambda F \\ M+\lambda F & N+\lambda G \end{pmatrix}$$

$$= \frac{\lambda^2(EG-F^2) + (LG+NE-2MF)\lambda + LN-M^2}{EG-F^2}$$

$$= \lambda^2 + 2H\lambda + K$$ where H = mean curvature, K = Gaussian "

Thus by Cayley-Hamilton's theorem

$$T^2 + 2HT + K = 0.$$

2. The portion of the ellipsoid above the xy-plane is the graph of $f(x,y) = c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

$$f(u,v) = c\sqrt{1-\frac{u^2}{a^2}-\frac{v^2}{b^2}}$$

$$\Rightarrow f_u = -\frac{c}{a^2}\cdot\frac{u}{\sqrt{1-\frac{u^2}{a^2}-\frac{v^2}{b^2}}}, \quad f_v = -\frac{c}{b^2}\frac{v}{\sqrt{1-\frac{u^2}{a^2}-\frac{v^2}{b^2}}}$$

$$\Rightarrow f_{uu} = -\frac{c}{a^2}\left\{\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-1/2} + u\left(-\tfrac{1}{2}\right)\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-3/2}\cdot\left(\frac{-2u}{a^2}\right)\right\}$$

$$= -\frac{c}{a^2}\left\{\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-1/2} + \frac{u^2}{a^2}\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-3/2}\right\}$$

$$= -\frac{c}{a^2}\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-3/2}\left\{1-\frac{u^2}{a^2}-\frac{v^2}{b^2}+\frac{u^2}{a^2}\right\}$$

$$= -\frac{c}{a^2}\left(1-\frac{v^2}{b^2}\right)\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-3/2}$$

Similarly, $f_{vv} = -\frac{c}{b^2}\left(1-\frac{u^2}{a^2}\right)\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-3/2}$,

$$f_{uv} = -\frac{c}{a^2b^2}uv\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-3/2}$$

Now, apply the formula $K = \dfrac{f_{uu}f_{vv}-f_{uv}^2}{(1+f_u^2+f_v^2)^2}$.

$$\text{Numerator} = \frac{c^2}{a^2b^2}\left(1-\frac{u^2}{a^2}\right)\left(1-\frac{v^2}{b^2}\right)\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-3}$$

$$-\frac{c^2}{a^4b^4}u^2v^2\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-3}$$

$$= \frac{c^2}{a^2b^2}\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-3}\left\{\left(1-\frac{u^2}{a^2}\right)\left(1-\frac{v^2}{b^2}\right)-\frac{u^2v^2}{a^2b^2}\right\}$$

$$= \frac{c^2}{a^2b^2}\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-2}$$

$$\text{Denominator} = \left\{1+\frac{c^2}{a^4}\cdot\frac{u^2}{1-\frac{u^2}{a^2}-\frac{v^2}{b^2}}+\frac{c^2}{b^4}\frac{v^2}{1-\frac{u^2}{a^2}-\frac{v^2}{b^2}}\right\}^2$$

$$= a^{-8}b^{-8}\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-2}\left\{\cancel{a^4b^4}+c^2u^2+c^2v^2+a^4b^4\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)\right\}^2$$

$$= a^{-8}b^{-8}\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-2}\left\{c^2u^2+c^2v^2+a^4b^4-b^2u^2-a^2v^2\right\}^2$$

$$= a^{-8}b^{-8}\left(1-\frac{u^2}{a^2}-\frac{v^2}{b^2}\right)^{-2}\left\{a^4b^4+(c^2-b^2)u^2+(c^2-a^2)v^2\right\}^2$$

$$= \cancel{a^{-8}b^{-8}}$$

$$\Rightarrow K = \frac{c^2a^6b^6}{\left(a^4b^4-(b^2-c^2)u^2-(a^2-c^2)v^2\right)^2} \longrightarrow (*)$$

Note: $\frac{u^2}{a^2}+\frac{v^2}{b^2} \le 1$. By smoothness of K $(*)$ is valid for $\frac{u^2}{a^2}+\frac{v^2}{b^2}=1$ too.

Let $s = \frac{u^2}{a^2}$, $t = \frac{v^2}{b^2}$. Then $s+t \le 1$ and $s, t \ge 0$.

The expression in the denominator of K

$$= \left(a^4b^4-(b^2-c^2)a^2s-(a^2-c^2)b^2t\right)^2$$

is minimum or maximum according as $a^4b^4-(b^2-c^2)a^2s-(a^2-c^2)b^2t$ is minimum or maximum subject to $s,t\ge 0$, $s+t\le 1$

Since $(a^2-c^2)>0$, $b^2-c^2>0$ we can simply (4)
say that the denominator max or min
according as $(a^2-c^2)b^2t + (b^2-c^2)a^2s$ is min
or max resp.

Thus K is min or max according as
$(a^2-c^2)b^2t + (b^2-c^2)a^2s$ is min or max resp
subject to $s,t \geq 0$, $s+t \leq 1$.

Since $(a^2-c^2)b^2t + (b^2-c^2)a^2s \geq 0$ it follows
that min. happens for $s=t=0$ i.e.
$u,v=0$.

This is a linear programing problem. The maximum
occurs at a vertex because along every line
in the triangle $s+t\leq 1$, $s,t\geq 0$ the
function $(a^2-c^2)b^2t + (b^2-c^2)a^2s$ is monotonic.

The vertex $(0,0)$ gives minimum.
Look at $(0,1)$ and $(1,0)$ and compare:

$(s,t) = (0,1)$: $(a^2-c^2)b^2 = a^2b^2 - b^2c^2$
$(s,t) = (1,0)$: $(b^2-c^2)a^2 = a^2b^2 - a^2c^2$

Since $a>b>c$ the maximum is at
$(s,t) = (1,0)$ i.e. $u=\pm a$, $v=0$

<u>Conclusion</u>: On the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$,
$a>b>c>0$, $K>0$ at all points and K is max.
at $(\pm a,0,0)$ and min. at $(0,0,\pm c)$

Note: Reflection in the xy- plane gives (5)
an isometry from the ~~[illegible]~~ upper half of the ellipsoid to the lower half. By Theorem Egregium or by ~~an~~ actual calculation we can calculate curvature ~~of~~ ~~at~~ ~~of~~ on the lower half of the ellipsoid. It follows that $K(x,y,z) = K(x,y,-z)$.
Hence, for K_{max}, K_{min} it is enough to look all $z \geq 0$.

3. Consider the parametrization

$$t \longmapsto (0,\ 2+\cos t,\ \sin t) \quad \text{of } C$$

To get a surface patch we restrict t to $(0, 2\pi)$.

Now, consider $\varphi(\theta,t) = ((2+\cos t)\cos\theta,\ (2+\cos t)\sin\theta,\ \sin t)$

$t, \theta \in (0, 2\pi)$. This is a surface patch.

$$\varphi_\theta = (-(2+\cos t)\sin\theta,\ (2+\cos t)\cos\theta,\ 0)$$

$$\varphi_t = (-\sin t\cos\theta,\ -\sin t\sin\theta,\ \cos t)$$

$$\varphi_\theta \times \varphi_t = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -(2+\cos t)\sin\theta & \cdots & \\ -\sin t\cos\theta & -\sin t\sin\theta & \cos t \end{vmatrix}$$

$$= (2+\cos t)\cos t\cos\theta\,\vec{i} + (2+\cos t)\cos t\sin\theta\,\vec{j} + (2+\cos t)\sin t\,\vec{k}$$

Since $2+\cos t > 0$, $\vec{N} = (\cos t\cos\theta,\ \cos t\sin\theta,\ \sin t)$

$\varphi_{\theta\theta} = (-(2+\cos t)\cos\theta, -(2+\cos t)\sin\theta, 0)$

$\varphi_{\theta t} = (\sin t \sin\theta, \ -\sin t\cos\theta, \ 0)$

$\varphi_{tt} = (-\cos t\cos\theta, -\cos t\sin\theta, -\sin t)$

$E = \varphi_\theta \cdot \varphi_\theta = (2+\cos\theta)^2$

$F = \varphi_\theta \cdot \varphi_t = 0$

$G = \varphi_t \cdot \varphi_t = 1$

$L = \varphi_{\theta\theta} \cdot \vec{N} = -(2+\cos t)\cos\theta \cdot \cos t\cos\theta$
$\qquad - (2+\cos t)\sin\theta \cdot \cos t \sin\theta$

$= -(2+\cos t)\cos t$

$M = \varphi_{\theta t} \cdot \vec{N} = \sin t\sin\theta \cdot \cos t\cos\theta - \sin t\cos\theta \cdot \cos t\sin\theta$
$= 0$

$N = -\cos t\cos\theta \cdot \cos t\cos\theta - \cos t\sin\theta \cdot \cos t\sin\theta - \sin t \cdot \sin t$
$= -(\cos^2 t\cos^2\theta + \cos^2 t\sin^2\theta + \sin^2 t) = -1$

Hence, w.r.t $\{\varphi_\theta, \varphi_t\}$ the Gauss map has matrix

$$-\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{(2+\cos\theta)^2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (2+\cos t)\cos t & 0 \\ 0 & 1 \end{pmatrix}$$

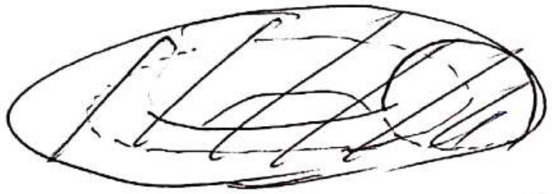
$$= \begin{pmatrix} \frac{\cos t}{2+\cos t} & 0 \\ 0 & 1 \end{pmatrix}$$

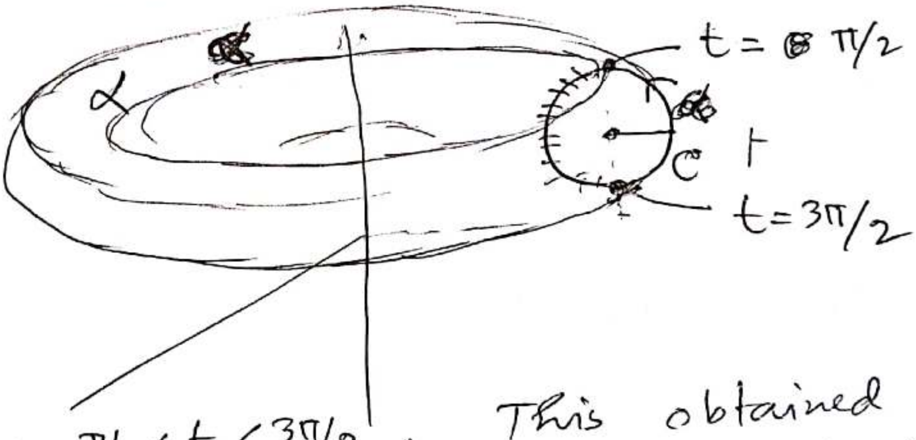
By smoothness of K, at all $\varphi(\theta, t)$,

$\cancel{K = \frac{\cos t}{2+\cos t}}$ $\qquad K = \dfrac{\cos t}{2+\cos t}.$

$K > 0$, $K = 0$ or $K < 0$ according as
$\cos t > 0$, $\cos t = 0$, $\cos t < 0$ $\quad t \in [0, 2\pi]$.

$\cos t = 0 \iff t = \pi/2, 3\pi/2$ This happens on α drawn below and its reflection in the xy-plane.





$\cos t < 0$ for $\pi/2 < t < 3\pi/2$. This obtained by revolving the shaded portion of C about the z-axis. Revolving the complementary portion of C we get all points with negative curvature.

Mean curvature $= -\frac{1}{2}\cos t\,(2+\cos t)$.

4) (i) $S_1 : x^2+y^2 = 1$

Take the line C: $y=1, x=0$ in the yz-plane with parametrization $t \mapsto (0,1,t)$

Then $\varphi(\theta,t) = (\cos\theta, \sin\theta, t)$ is a surface patch on S_1 for suitable θ-intervals.

$\varphi_\theta = (-\sin\theta, \cos\theta, 0)$

$\varphi_t = (0, 0, 1)$

$\varphi_{\theta\theta} = (-\cos\theta, -\sin\theta, 0)$

$\varphi_{\theta t} = (0,0,0)$

$\varphi_{tt} = (0,0,0)$

$$\varphi_\theta \times \varphi_t = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$$

$E = \varphi_\theta \cdot \varphi_\theta = 1$, $F = \varphi_\theta \cdot \varphi_t = 0$, $G = \varphi_t \cdot \varphi_t = 1$

$L = \varphi_{\theta\theta} \cdot \vec{N} = -1$, $M = \varphi_{\theta t} \cdot \vec{N} = 0$

$N = \varphi_{tt} \cdot \vec{N} = 0$

Hence, w.r.t $\{\varphi_\theta, \varphi_t\}$ the matrix of the Gauss map is

$$-\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence, $K = 1$, $H = -\frac{1}{2}$

4.(ii) The cone is obtained by revolving

$C : y = z \;, \; z > 0$

about the z-axis.

Choose parametrization $t \mapsto \left(0, \frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}\right)$

(unit speed) for C.

Rest is left as exercise.

5. Easy.

6. (i) Let $\varphi : \mathcal{U} \to S$ be a surface patch

and $S \xrightarrow{h} S'$ be a rigid motion.

Check : $\psi = h \circ \varphi : \mathcal{U} \to S'$ is a surface patch.

Note : • $\psi_u = Dh(\varphi_u)$

• $\psi_v = Dh(\varphi_v)$

• Dh is the same orthogonal linear map. Call it $\mathcal{L}$

$$\Rightarrow \left.\begin{array}{l} \psi_u = \mathcal{L}(\varphi_u) \\ \psi_v = \mathcal{L}(\varphi_v) \end{array}\right\} \Rightarrow \begin{array}{l} \psi_{uu} = \mathcal{L}(\varphi_{uu}) \\ \psi_{ut} = \mathcal{L}(\varphi_{ut}) \\ \psi_{tt} = \mathcal{L}(\varphi_{tt}) \end{array}$$

Note : Moreover, h being a rigid motion $\det \mathcal{L} > 0$.

Check : $\Rightarrow \mathcal{L}(\vec{v_1} \times \vec{v_2}) = \mathcal{L}\vec{v_1} \times \mathcal{L}\vec{v_2} \quad \forall \vec{v_1}, \vec{v_2} \in \mathbb{R}^3$

Rest is immediate.

(ii) Left as exercise : $h(x,y,z) = c(x,y,z)$

$Dh = h$ etc.

7. Optional exercise.
We will remark on this later.

8. If we have a curve $C: t \mapsto (0, y(t), z(t))$ in the yz-plane the surface obtained by revolving C about the z-axis has curvature

$$K = -\frac{y''}{y}$$

at any point if $y'^2 + z'^2 = 1$.

Thus to get the required surface we need to solve $-\frac{y''}{y} = -1$ i.e. $y'' = y$

and $x'^2 + y'^2 = 1$.

$y'' = y$ has solution $y(t) = ae^t + be^{-t}$ (a, b arbitrary constant).

$$\Rightarrow \quad x'^2 + (ae^t + be^{-t})^2 = 1$$

Solving this in exact form may not be possible.
Try the special case $a = 1$, $b = 0$:

$$x'^2 + e^{2t} = 1 \Rightarrow x' = \pm\sqrt{1 - e^{2t}}$$

Consider, $x' = \sqrt{1 - e^{2t}}$. This makes sense iff $t \leq 0$.
In that case $x = \int \sqrt{1 - e^{2t}}\, dt$

(Check) $= c + \sqrt{1 - e^{2t}} - \log\left(e^{-t} + \sqrt{e^{-2t} - 1}\right)$.