

Homework 10

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* means difficult or important
** Optional / too difficult.

0)* Solve all problem/exercises mentioned in class.
1)* If S_1 is orientable and S_2 is diffeomorphic to S_1 , then S_2 is also orientable.

** 2) Suppose $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ is a smooth function and $c \in \text{Im}(f)$. Let $S = f^{-1}(c)$.

Assume $\text{grad}(f) \neq 0$ on S . We want to show that S is a regular surface. The idea is to solve $f(x, y, z) = c$ for ~~one~~ one of the variables; e.g. if $f = x^2 + y^2 + z^2$, $c = 1$ then we can solve $z = \pm \sqrt{1 - x^2 - y^2}$. Like the example shows the solution may not be unique but if we get a solution say $z = g(x, y)$ then S is the graph of g and hence a smooth surface.

Given $p \in S$ assume $f_z \neq 0$. We ~~can~~

claim we can solve $f(x, y, z) = c$ as $z = g(x, y)$

"near p ". If $f_z = 0$ either $f_x \neq 0$ or $f_y \neq 0$.

If $f_x \neq 0$ then "near p " we will have $x = h(y, z)$

etc. We deal with the case $f_z \neq 0$.

Consider $F(x, y, z) = (x, y, f(x, y, z))$,

$F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$.

⊗ check that $JF = \begin{pmatrix} 1 & 0 & f_x \\ 0 & 1 & f_y \\ 0 & 0 & f_z \end{pmatrix}$. (2)

Hence $\det JF(p) \neq 0$. By Inverse function Theorem there is an open set $W \subseteq \mathbb{R}^3$ such that $W \ni p$ and $F: W \rightarrow F(W)$ is a diffeomorphism. Let $G: F(W) \rightarrow W$ be the inverse of F . Let $G(x, y, z) = (\varphi_1(x, y, z), \varphi_2(x, y, z), \varphi_3(x, y, z))$.

For any $(x, y, z) \in W \cap S$ we have $G \circ F(x, y, z) = (x, y, z)$ since $G \circ F = \text{Id}_W$

$$\Rightarrow G(x, y, f(x, y, z)) = (x, y, z)$$

$$\Rightarrow G(x, y, c) = (x, y, z)$$

In particular, $z = \varphi_3(x, y, c)$.

Define $g(x, y) = \varphi_3(x, y, c)$.

⊗⊗ Determine the domain of g , say V . Find a surface patch using this.

This shows that every point of S falls in a surface patch. Thus S is a regular surface. Check all conditions for allowable surface patches carefully.

3) Check that if $z = g(y)$ is a smooth ③
curve in the yz -plane not touching or crossing
the z -axis then revolving it about the z -axis
we get an orientable surface.

4)* If S is a plane ~~ax~~ say $ax + by + cz = d$
then show that $\forall p \in S$, $T_p S$ is the plane
 $ax + by + cz = 0$.

5)* (A) Consider the map $f: S \rightarrow \mathbb{R}^2$ where
 $S: z = x^2 + y^2$ and $f(x, y, z) = (x, y)$. Let $p =$
 $(1, 1, 2) \in S$. (i) Find $T_p S$

(ii) \mathbb{R}^2 can be thought of as the xy -plane
in \mathbb{R}^3 . Let $q = f(p)$. Determine the
map $Df_p: T_p S \rightarrow T_q \mathbb{R}^2$. In other words
find any basis of ~~the~~ $T_p S$, say (\vec{v}_1, \vec{v}_2) and
evaluate $Df_p(\vec{v}_1), Df_p(\vec{v}_2)$

(B)* Do the same for $S = S^2: x^2 + y^2 + z^2 = 1$,
 $p = (0, 0, 1)$, $f: S^2 \rightarrow \mathbb{R}^2 (x, y, z) \mapsto (y, z)$.

(C)* Do the same for $f: S_1 \rightarrow S_2$ where
 $S_1: z = \sqrt{x^2 + y^2}, z > 0$, $S_2: x^2 + y^2 = 1$

$$f(x, y, z) = \left(\frac{x}{z}, \frac{y}{z}, z \right), \quad p = (1, 0, 1) \text{ i.e. } \textcircled{4}$$

(i) Find $T_p S_1$, $T_{f(p)} S_2$ and describe

$$Df_p: T_p S_1 \rightarrow T_{f(p)} S_2.$$

(ii) Show that Df is an isomorphism at all points of S_1 . See next problem.

4. A map $f: S_1 \rightarrow S_2$ is called a local diffeomorphism if $\forall p \in S_1$ there is an open set $U \subseteq S_1$, $p \in U$ such that $f(U) \subseteq S_2$ is open and $f: U \rightarrow f(U)$ is a diffeomorphism.

(i) If f is a local diffeomorphism then show that Df_p is an isomorphism $\forall p \in S_1$.

(ii)** Prove the converse also: If Df_p is an isomorphism $\forall p \in S_1$ then f is a local diffeomorphism.

5. (A) Consider the eight surface patches on S^2 obtained by projecting points on the coordinate planes. Show that the transition of patches have positive Jacobian determinant.

(B)* Cover the cylinder $S: x^2 + y^2 = 1$ by two patches and check ~~the~~ such that the transition of patches has positive Jacobian determinant.