

Solution to HW 10

(1)

1) Suppose $f: S_1 \rightarrow S_2$ is a diffeomorphism.

Suppose $\{\varphi_i: U_i \subseteq \mathbb{R}^2 \rightarrow S_1\}$ is a collection of allowable surface patches such that $\det(J(\varphi_i^{-1}\varphi_j)) > 0$ whenever $\varphi_i(U_i) \cap \varphi_j(U_j) \neq \emptyset$ on the respective domains of $\varphi_i^{-1}\varphi_j$'s.

Let $\gamma_i = f \circ \varphi_i: U_i \rightarrow S_2$

Claim 1: $\{\gamma_i: U_i \rightarrow S_2\}$ is a collection of allowable surface patches.

Clearly γ_i is smooth since φ_i, f are smooth for each i . $f: \varphi_i(U_i) \rightarrow f(\varphi_i(U_i))$ is a homeomorphism since f is a diffeomorphism.

Thus $\gamma_i: U_i \rightarrow \gamma_i(U_i)$ is a homeomorphism.

Now, $\gamma_{iu} = Df(\varphi_{iu})$
 $\gamma_{iv} = Df(\varphi_{iv})$

We know that diffeomorphism induces linear isomorphism between tangent spaces. Since $\varphi_{iu}, \varphi_{iv}$ are linearly independent so are γ_{iu}, γ_{iv} . Hence, $\gamma_{iu} \times \gamma_{iv} \neq \vec{0}$.

Claim 2: $\det(J(\gamma_i^{-1}\gamma_j)) > 0$. This is clear:

Note that $\gamma_i^{-1} \circ \gamma_j = (f \circ \varphi_i)^{-1} \circ (f \circ \varphi_j) = \varphi_i^{-1} \circ \varphi_j$.

② This is left for interested people. ②

3) We know that the surface has patches of the form

$$\varphi_{ab}: \begin{matrix} (a, \frac{b}{2\pi}) \times I \rightarrow S \\ (\theta, t) \mapsto (t \cos \theta, t \sin \theta, \gamma(t)) \end{matrix}$$

where $I = \text{domain of } \gamma$, $\& b-a \leq 2\pi$.

~~Let~~ Let $U_{a,b} = (a,b) \times I$.

Now, two such surface patches intersect iff $(a,b) \cap (c,d) \neq \emptyset$.

check that the transition maps are identity.

4) S is the level surface $f(x,y,z) = d$ where $f = ax + by + cz$. Now, $\text{grad}(f) = (a, b, c)$ etc.

~~Hence~~
5) (A) $S: x^2 + y^2 - z = 0$
Let $f = x^2 + y^2 - z$. $\text{grad}(f) = (2x, 2y, -1)$

$$\Rightarrow \text{grad}(f)_{(1,1,2)} = (2, 2, -1)$$

Thus $T_p S: 2x + 2y - z = 0$

Let $\vec{u} = (1, 0, 2)$, $\vec{v} = (0, 1, 2)$. Then

~~The~~ $\{\vec{u}, \vec{v}\}$ is a basis of $T_p S$.

Now, f is the restriction of the smooth map $F: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ which has $(x, y, z) \mapsto (x, y)$ (3)

Jacobian $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ at all points.

Hence, if α is any curve in S passing through $p = (1, 1, 2)$, $\alpha'(p) = \vec{u}$ then

$$Df_p(\vec{u}) = DF_p(\vec{u}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Similarly, $Df_p(\vec{v}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(B) Very similar to (A).

(C) In this case let $U \subseteq \mathbb{R}^3$ be the open set $\{(x, y, z) : z > 0\}$. Then $S \subseteq U$ and f is the restriction of

$$F: U \rightarrow \mathbb{R}^2$$

$$(x, y, z) \mapsto \left(\frac{x}{z}, \frac{y}{z}, z\right)$$

Now, do as in (A).

6.(i) Df_p depends on the values of f near p , i.e. values of f on any open set $U \ni p$.

If $f: \mathcal{U} \rightarrow f(\mathcal{U})$ is a diffeomorphism then (4)

$g := f^{-1}: f(\mathcal{U}) \rightarrow \mathcal{U}$ is smooth.

Now, $g \circ f \circ g^{-1}: \mathcal{U} \rightarrow \mathcal{U}$ is the identity.

Using chain rule

$$Dg_{f(p)} \circ Df_p = \text{Id} : T_p \mathcal{S}_1 \rightarrow T_p \mathcal{S}_1$$

Note: $T_p \mathcal{U} = T_p \mathcal{S}_1$.

Thus Df_p is injective. Since Df_p

Similarly $Df_p \circ Dg_{f(p)}: T_{f(p)} \mathcal{S}_2 \rightarrow T_{f(p)} \mathcal{S}_2$ is identity. Thus $Dg_{f(p)} = Df_p^{-1}$.

(ii) Let $\varphi: \mathcal{U} \rightarrow \mathcal{S}_1$ be a surface patch, $p \in \varphi(\mathcal{U})$ such that there is a surface patch $\psi: \mathcal{V} \rightarrow \mathcal{S}_2$ where $f \circ \varphi(\mathcal{U}) \subseteq \psi(\mathcal{V})$. Let $\tilde{p} = \psi^{-1}(p)$.

Let $F = \psi^{-1} \circ f \circ \varphi: \mathcal{U} \rightarrow \mathcal{V}$.

Since φ, ψ are diffeomorphisms $DF_x: T_x \mathcal{U} \rightarrow T_{F(x)} \mathcal{V}$ is an isomorphism or $\det(JF_x) \neq 0$.

Now, F being smooth, the map $h: \mathcal{U} \rightarrow \mathbb{R}$ defined by $(x, v) \mapsto \det JF_{(x,v)}$ is a smooth function. Since $h(x) \neq 0$ there is an open

By inverse function theorem there is an open set $\mathcal{U}_1 \subseteq \mathcal{U}$ such that $F: \mathcal{U}_1 \rightarrow F(\mathcal{U}_1)$ is a diffeomorphism, $p \in \mathcal{U}_1$, $F(\mathcal{U}_1) \subseteq \mathcal{V}$ open.

check: $\psi \circ F: \mathcal{U}_1 \rightarrow f \circ \varphi(\mathcal{U}_1)$ is a diffeomorphism.

(7) Do it yourself.