REAL
ANALYSIS

MTH102

Edited by

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Contents

Chapter 1

The set of $\mathbb{R}$ and some of it's properties

§1.1 The set of $\mathbb{R}$

Definition 1.1.1: Completeness Axiom

Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. In other words, *sup* S exists and is a real number

Corollary 1.1.1

Every nonempty subset S of $\mathbb{R}$ that is bounded below has a greatest lower bound. In other words, inf S exists and is a real number.

Proof. Let S be a set which is bounded below i.e. $\forall s \in S$, $s \geq m$ for some $m \in \mathbb{R}$ .Take the set $-S = \{-s \mid s \in S\}$. Therefor $\forall u \in -S \;\Rightarrow \{u \leq -m\}$ or

$$\Rightarrow \;\; u \leq sup(-S) \leq -m \;\; \{\text{completeness axiom}\}$$

$$\Rightarrow \;\; -s \leq sup(-S) \leq -m$$

$$\Rightarrow \;\; m \leq -sup(-S) \leq s$$

Let $\exists\ \lambda \in \mathbb{R}$ such that $-sup(-S) \leq \lambda \leq s$ which implies $-s \leq -\lambda \leq sup(-S)$. Since, $-\lambda$ cannot be the supremum of S.

Therefore, $\boxed{\text{inf(S) = -sup(-S)}}$.

■

Theorem 1.1.1: Archemedian Property

If a $\geq$ 0 and b $\geq$ 0, then $\exists\ n \in \mathbb{N}$ such that $na \geq b$. Or in other words we can say that the set of natural numbers is not bounded above.

Proof. Assume the Archimedean property fails. Then there exist a $\geq$ 0 and b $\geq$ 0 such that na $\leq$ b $\forall\ n \in \mathbb{N}$. In particular, b is an upper bound for the set $S = \{na \mid n \in \mathbb{N}\}$. Let $s_0 = supS$; this is where we are using the completeness axiom. Since a ≥ 0, we have $s_0 \leq s_0 + a$, so s_0-$a \leq s_0$. Since s_0 is the least upper bound for S, $s_0 - a$ cannot be an upper bound for S. It follows that $s_0 - a \leq n_0 a$ for some $n_0 \in \mathbb{N}$ This implies $s_0 \leq (n_0 + 1)a$. Since $(n_0 + 1)a$ is in S, s_0 is not an upper bound for S and we have reached a contradiction. Our assumption that the Archimedean property fails was wrong. ∎

Chapter 2

Sequences

§2.1 Convergence

Definition 2.1.1: Convergence

A sequence $\{s_n\}$ is said to *converge* to a real number "s" provided that for each $\epsilon > 0$ there exist as number N such that

$$n > N \text{ implies } |s_n - s| < \epsilon \tag{2.1}$$

If (s_n) converges to s, we will write $\lim_{n\to\infty} s_n = s$, or $s_n \to s$. The number s is called the limit of the sequence (s_n) . A sequence that does not converge to some real number is said to diverge (Is it?)

Problem 1. *Let (s_n) be a sequence of nonnegative real numbers and suppose $s = \lim s_n$. Note $s \geq 0$. Prove $\lim \sqrt[2]{s_n} = \sqrt[2]{s}$*

Solution. **Case 1:**$(s > 0)$ Let $\epsilon > 0$. Since $\lim s_n = s \;\Rightarrow\; \exists\, N$ such that

$$n > N \text{ implies } |s_n - s| < \epsilon \tag{2.2}$$

Now, $n > N$ implies

$$|\sqrt{s_n} - \sqrt{s}| = \frac{|s_n - s|}{\sqrt{s_n} + \sqrt{s}} \leq \frac{|s_n - s|}{\sqrt{s}} < \frac{\sqrt{s}\epsilon}{\sqrt{s}} = \epsilon$$

Case 2:$(s = 0)$ Since $s_n > 0 \Rightarrow |s_n - 0| < \epsilon \;\forall\; \epsilon > 0$. Take ϵ to be ϵ^2 .

$$\Rightarrow n > N \text{ implies } |s_n| < \epsilon^2 \Rightarrow |\sqrt{s_n}| < \epsilon$$
$$\Rightarrow |\sqrt{s_n} - 0| < \epsilon$$

So, $\lim \sqrt{s_n} = 0$ ■

Problem 2. *Prove that:*

1. $\lim[\sqrt{n^2+1} - n] = 0$

2. $\lim[\sqrt{4n^2+n}-2n] = 1/4$

Solution. I think it's enough to discuss the strategy because the reader should be able to proceed backwards.

1. Since, $\sqrt{n^2+1}-n > 0$ we can simply remove the modulus sign and write $\sqrt{n^2+1}-n < \epsilon$
$$\Rightarrow n^2+1 < (\epsilon+n)^2$$
$$\Rightarrow n^2+1 < \epsilon^2+n^2+2n\epsilon$$
$$\Rightarrow \frac{1-\epsilon^2}{2\epsilon} < n$$
So, take your $N = \frac{1-\epsilon^2}{2\epsilon}$ and proceed backwards

2. We can show that $\frac{1}{4} > \sqrt{4n^2+n}-2n$ by simply assuming the inequality and we'll get the result that $1/4 > 0$ which is indeed true or if we assume other inequality it'll lead to the contradiction.
Since, $\frac{1}{4} - (\sqrt{4n^2+n}-2n) > 0$ we can write
$$\Rightarrow \frac{1}{4} - (\sqrt{4n^2+n}-2n) < \epsilon$$
$$\Rightarrow \frac{1}{4} - \epsilon + 2n < \sqrt{4n^2+n}$$
To square both the sides $\frac{1}{4}-\epsilon+2n > 0,\ \forall n \in \mathbb{N}$ which is if $\epsilon < 9/4$. Squaring both sides and cancelling the similar terms we get
$$\Rightarrow (\epsilon-1/4)^2 < n(1-4(\epsilon-1/4)^2)$$
Divide both side by $(1-4(\epsilon-1/4)^2)$ for that $1-4(\epsilon-1/4)^2 > 0$ or $\epsilon < 3/4$ which also satisfies the above condition of $\epsilon < 9/4$. Since,we have to prove it for small enough ϵ. So, for bigger epsilon it's automatically true i.e. if we prove for $\epsilon < 3/4$ then it is true for $\epsilon > 9/4$. So,
$$\Rightarrow \frac{(\epsilon-1/4)^2}{(1-4(\epsilon-1/4)^2)} < n$$
Hence, to write a formal proof take your $N = \frac{(\epsilon-1/4)^2}{(1-4(\epsilon-1/4)^2)}$

■

§2.2 Limit Theorem for Sequences

Definition 2.2.1

A sequence (s_n) is said to be bounded if there exist a real number M, such that $|s_n| \leq M\ ,\forall n \in \mathbb{N}$

Geometrically, this means that we can find an interval $[-M, M]$ that contains every term in the sequence (x_n).

Theorem 2.2.1

Convergent sequences are bounded

Proof. Let (s_n) be a convergent sequence of real numbers, let $\lim s_n = s$. Applying 2.1.1 with $\epsilon = 1$ we obtain N in $\mathbb{N}$ such that

$$\Rightarrow n > N \text{ implies } |s_n - s| < 1$$

From triangular inequality it implies $|s_n| < |s|+1$ for $n > N$. Take M $= max\{|s_1|, |s_2|, |s_3|, \ldots |s|+1\}$ .Then we have $|s_n| \leq M$ for all $n \in N$, so (s_n) is bounded.

The choice of ϵ is arbitrary ■

Theorem 2.2.2

If the sequence (s_n) converges to s and k is in $\mathbb{R}$, then the sequence (ks_n) converges to (ks). That is, $\lim(ks_n) = k \cdot \lim s_n$.

Theorem 2.2.3

If s_n converges to s and t_n converges to t, then $(s_n + t_n)$ converges to $(s + t)$

$$\lim(s_n + t_n) = (s + t)$$

Theorem 2.2.4

If s_n converges to s and t_n converges to t, then $(s_n \cdot t_n)$ converges to $(s \cdot t)$

$$\lim(s_n \cdot t_n) = (s \cdot t)$$

The theorem can be proved using the identity $(a + b)^2 = a^2 + b^2 + 2ab$ take $(s_n + t_n)$ *and* $(s_n - t_n)$ and proceed. (Wait did I prove $\lim(a_n)^2 = a^2$ if $a_n \to a$. Well, if not then it's easy to prove.)

Theorem 2.2.5

If s_n converges to s. Then $1/s_n$ converges to $1/s$ for $(s \neq 0)$

Proof. We begin by observing that

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s_n - s|}{|s_n s|}$$

Because $(s_n) \to s$, we can make the preceding numerator as small as we like by choosing n large. The problem comes in that we need a worst-case estimate on the size of $1/(|s||s_n|)$. Because the s_n terms are in the denominator, we are no longer interested in an upper bound on $|s_n|$ but rather in an inequality of the form $|s_n| \geq \delta > 0$. This will then lead to a bound on the size of $1/(|s||s_n|)$. The trick is to look far enough out into the sequence (s_n) so that the terms are closer to s than they are to 0. Consider the particular value $\epsilon = |s|/2$. Because $(s_n) \to s$, there exists an N_1 such that $|s_n - s| < |s|/2$ for all $n \geq N_1$. This implies $|s_n| > |s|/2$. Next, choose N_2 so that $n \geq N_2$ implies $|s_n - s| < |s|$

$$|s_n - s| < \frac{\epsilon \cdot s^2}{2}$$

Finally, if we let $N = max\{N_1, N_2\}$, then $n \geq N$ implies.

$$\left| \frac{1}{s_n} - \frac{1}{s} \right| = |s_n - s| \frac{1}{|s_n s|} < \frac{\epsilon s^2}{2} \frac{1}{|s| \frac{|s|}{2}} = \epsilon$$

■

Theorem 2.2.6

If s_n converges to s and t_n converges to t (t $\neq 0$), then $\frac{s_n}{t_n}$ converges to $(\frac{s}{t})$

$$\lim \frac{s_n}{t_n} = \frac{s}{t}$$

Proof. The proof is trivial and is left as an exercise for the readers ■

Theorem 2.2.7

If $s_n < t_n$ then $\lim s_n \leq \lim t_n$.

Proof. Let $s = \lim s_n$ and $t = \lim t_n$. Let $h_n = t_n - s_n > 0$. So, $h = \lim h_n \geq 0$ (Why?). Let's take $h < 0 \Rightarrow -h > 0$. So, there exists a N such that $n > N$ implies $|h_n - h| < -h \Rightarrow h_n < 0$. Contradiction! So, $h \geq 0 \Rightarrow t - s \geq 0$. ■

Theorem 2.2.8: (Basic Examples)

a. $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for $p > 0$.

b. $\lim_{n\to\infty} a^n = 0$ if $|a| < 1$.

c. $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

d. $\lim_{n\to\infty} a^{\frac{1}{n}} = 1,\ a > 0$.

Problem 3. *For a sequence (s_n) of positive real numbers, we have $\lim s_n = +\infty$ if and only if $\lim(\frac{1}{s_n}) = 0$.*

Proof. We need to prove that

$$\lim s_n = +\infty \ \Rightarrow \ \lim(1/s_n) = 0$$

and

$$\lim(1/s_n) = 0 \ \Rightarrow \ \lim s_n = +\infty$$

1. Since $\lim s_n = +\infty$ so for every $n > N$ there exist a M such that $s_n > M$. Take $M = 1/\epsilon,\ \epsilon > 0$

$$\Rightarrow \quad s_n > M = 1/\epsilon$$
$$\Rightarrow \quad \epsilon > \frac{1}{s_n} > 0$$
$$\Rightarrow \quad \left|\frac{1}{s_n} - 0\right| < \epsilon$$

2. Workout the above proof backwards by assuming $\epsilon = 1/M$

■

Problem 4. *Let $s_1 = 1$ and $s_{n+1} = \sqrt{1+s_n}$.*
Assume that the sequence converge. Prove that the sequence converges to $\frac{1}{2}(1+\sqrt{5})$

Proof. Let the sequence converge to s.So, as $n \to \infty$ $s_{n+1} = s_n = s$ therefore

$$\Rightarrow s_{n+1} = \sqrt{1+s_n}$$
$$\Rightarrow s_{n+1}^2 - s_n - 1 = 0$$
$$\Rightarrow s^2 - s - 1 = 0 \text{ as } n \to \infty$$

So, $s = \frac{1}{2}(1+\sqrt{5})$ ■

In this problem we have assumed that the limit exist

Theorem 2.2.9

Let $\{a_n\}$ be a sequence of positive numbers such that $\lim_{n\to\infty} a_n = L$. Prove that $\lim_{n\to\infty} \sqrt[n]{(a_1 a_2 \cdots a_n)} = L$

Theorem 2.2.10

If $\lim\left|\frac{a_{n+1}}{a_n}\right|$ exists [and equals L], then $\lim(a_n)^{\frac{1}{n}}$ exists [and equals L].
Also, deduce $\lim_{n\to\infty} \frac{n}{(n!)^{\frac{1}{n}}}$

Proof. Define the sequence $\{b_n\}$ by $b_1 = a_1$ and $b_n = \frac{a_n}{a_{n-1}}$ for $n \geq 2$. Since $\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = L$, we have $\lim_{n\to\infty} b_n = L$. Note that $a_n = (b_1 b_2 \dots b_n)$. Applying the above Theorem to the sequence b_n, we get :

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} (b_1 b_2 \dots b_n)^{\frac{1}{n}} = L$$

Now, let $a_n = \frac{n^n}{n!}$. Note that

$$\lim_{n\to\infty} \frac{a_n+1}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^{(n+1)}}{(n+1)!}}{\frac{n^n}{n!}} = \lim_{n\to\infty} \frac{\frac{(n+1)^{(n+1)}}{(n+1)}}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

By the conclusion above, we have:

$$\lim_{n\to\infty} \frac{n^n}{(n!)^{\frac{1}{n}}} = \lim_{n\to\infty} a_n^{\frac{1}{n}} = e$$

■

§2.3 Monotone and Cauchy Sequences

Theorem 2.3.1

All bounded monotone sequences are convergent

Proof. I'll prove it for monotonically decreasing sequences
Let (s_n) be a bounded decreasing sequence i.e. $\forall\ n \in \mathbb{N}\ [s_{n+1} < s_n]$ & $[\ |s_n| < M]$ for some $M \in \mathbb{R}$.

Since, $|s_n| < M \;\Rightarrow\; -M < s_n < M$. Applying Corollary 1.1.1 there exist a $\lambda \in \mathbb{R}$ such that $-M < \lambda < s_n$ for all $n \in \mathbb{R}$. Since, λ is the greatest lower-bound, therefore for any $\epsilon > 0$ there exist a N such that $s_N < \lambda + \epsilon$. Also, the sequence is deceasing so $\lambda - \epsilon < \lambda \leq s_{n+1} \leq s_n \leq \lambda < \lambda + \epsilon$

$$\Rightarrow\; -\epsilon < s_n + \lambda < \epsilon$$

$$\Rightarrow\; |s_n - \lambda| < \epsilon$$

So, inf(s_n)=$\lim s_n$ in case of decreasing sequences

■

Definition 2.3.1: $\limsup' s$ **&** $\liminf' s$

For some sequence s_n we define:

$$\limsup s_n = \lim_{N\to\infty} \sup\{s_n \ :\ n > N\}$$

and

$$\liminf s_n = \lim_{N\to\infty} \inf\{s_n \ :\ n > N\}$$

Problem 5. *Calculate the* $\limsup a_n$ *and* $\liminf a_n$ *for* $a_n = (-1)^n \frac{(n+5)}{n}$

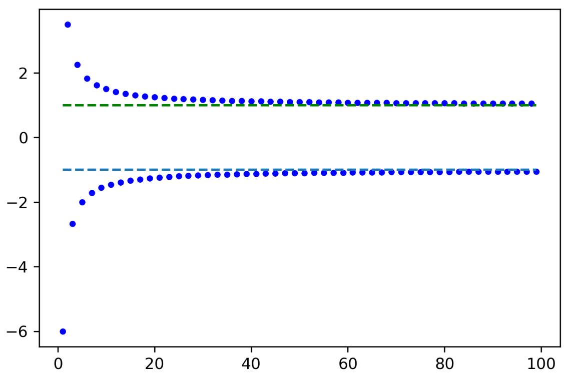


Figure 2.1: $(-1)^n \frac{(n+5)}{n}$ v/s n

Solution. Since in the set of subsequences of a_n the least element will be the $\liminf$ and the greatest term would be $\limsup$ so. Since the -ve values are less than the +ve ones and in case of $-1 - \frac{5}{n}$ the least value for n gives the least of all so,we can say the first negative term of each subsequence will be the element of subsequential infimimum.Therefore, $\lim(-1 - \frac{5}{n}) = -1$ is the $\liminf$. Similar argument for $\limsup$ as the greatest of all will be sup of each subsequence and the least value of n will give the largest of all. So, the sequence $s_n = \frac{(n+5)}{n}$ will be sequence of subsequential supremum.And $\lim \frac{(n+5)}{n} = 1$. ■

Theorem 2.3.2

Let (s_n) be a sequence $\mathbb{R}$

(i) If $\lim s_n$ is defined then:

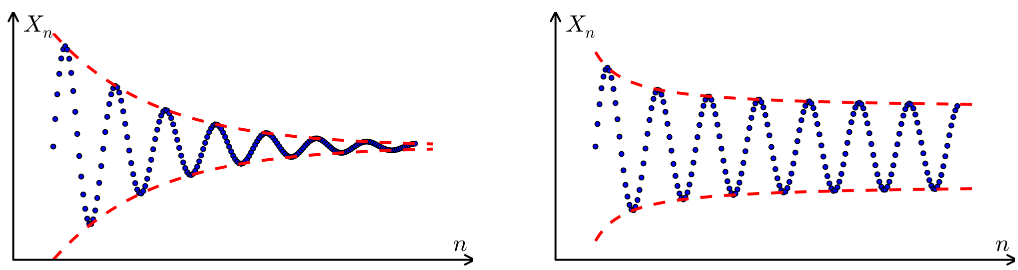
$$\liminf s_n = \limsup s_n = \lim s_n$$

(ii) If $\liminf s_n = \limsup s_n$ then $\lim s_n$ is defined and $\liminf s_n = \limsup s_n = \lim s_n$

Definition 2.3.2: *CAUCHY SEQUENCE*

[5] A sequence (s_n) of real numbers is said to be Cauchy if:
for each $\epsilon > 0$ there exist a N such :

$$\forall\ n, m > N \textit{ implies } |s_n - s_m| < \epsilon$$



(a) The plot of a Cauchy sequence (x_n), shown in blue, as (x_n) versus n

(b) A sequence that is not Cauchy.But is bounded

Figure 2.2: Illustartion for cauchy sequences

Lemma 2.3.1

Convergent sequence are Cauchy

Lemma 2.3.2

Cauchy sequence are bounded

Theorem 2.3.3

A sequence is a convergent sequence if and only if it is a Cauchy sequence.

Problem 6. *Prove that* $|s_{n+1} - s_n| < 2^{-n}$ *is cauchy. And hence convergent.*

Proof. We've to prove that $\forall \epsilon > 0$ there exist a N such that $\forall\ n, m > N$, $|s_m - s_n| < \epsilon$.

Take $m > n$ and let $m = n + k$. So, by triangular inequality

$$|s_m - s_n| \leq |s_m - s_{m-1}| + |s_{m-1} + s_{m-2}| + \ldots + |s_{n+1} + s_n| < \frac{1}{2^n} + \frac{1}{2^{n+1}} + \ldots + \frac{1}{2^m}$$

Since, m = n+k and if we add more +ve terms then we will get a G.P. with ratio $\frac{1}{2}$ which look like this:

$$\frac{1}{2^n} + \frac{1}{2^{n+1}} + \ldots + \frac{1}{2^m} = \frac{1}{2^n} + \frac{1}{2^{n+1}} + \ldots + \frac{1}{2^{n+k}} = \frac{1}{2^{n-1}}(1 - \frac{1}{2^k})$$

So,

$$|s_m - s_n| \leq |s_m - s_{m-1}| + |s_{m-1} + s_{m-2}| + \ldots + |s_{n+1} + s_n| < \frac{1}{2^{n-1}}(1 - \frac{1}{2^k}) < \frac{1}{2^{n-1}}$$

And applying Theorem 2.2.8 ,we can say $\frac{1}{2^{n-1}}$ is convergent. Hence, the same N will work for the sequence s_n. ■

We have not talked about Series / summation till now. Series, will be discussed in the next Section(3)

Problem 7. *Let* s_n *be an increasing sequence. Prove that* $\sigma_n = \frac{1}{n}(s_1 + s_2 + \ldots + s_n)$ *is an increasing sequence.*

Proof. Since, since $\sigma_1 < \sigma_2$. Let us suppose $\sigma_n > \sigma_{n-1}$. Now, we have to prove that it is true for the term σ_{n+1}.

Suppose, it's not true i.e. $\sigma_{n+1} < \sigma_n$

$$\frac{1}{n+1}(s_1 + s_2 + \ldots + s_{n+1}) < \frac{1}{n}(s_1 + s_2 + \ldots + s_n)$$

If we further simplify the inequality, we get $ns_{n+1} < (s_1 + s_2 + \ldots + s_n)$ which is false. So, the assumption was false. Hence, $\sigma_{n+1} > \sigma_n$. ■

Problem 8. *Define* $x_1 = 2$ *and:*

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

Prove that the $\lim_{n\to\infty} x_n = \sqrt{2}$.

Proof. Observer that $x_2 < x_1 = 2$. So, assume $x_{n+1} < x_n < x_{n-1} \ldots < x_1 = 2$.
Suppose $x_{n+2} > x_{n+1}$ (it would lead to a contradiction), it implies:

$$\Rightarrow \frac{1}{2}\left(x_{n+1} + \frac{2}{x_{n+1}}\right) > x_{n+1}$$

$$\Rightarrow (x_{n+1})^2 < 2$$

Substitute $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$ and solve. We get:

$$({x_n}^2 - 2)^2 < 0$$

which is a contradiction. So, the series is monotonically decreasing and bounded(because each element is greater than zero and less that two). Hence, convergent. To prove $\lim_{n\to\infty} x_n = \sqrt{2}$, put $x_{n+1} = x_n = x$ in the definition.($\because \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1}$) ■

§2.4 Subsequences

An unformal definition of subsequence: A **Subsequence** is a sequence that can be derived from another sequence by deleting some or no elements without changing the order of the remaining elements

Theorem 2.4.1

If the sequence (s_n) converges, then every subsequence converges to the same limit.

Theorem 2.4.2

Every sequence (s_n) has a monotonic subsequence.

Theorem 2.4.3: Bolzano-Weierstrass Theorem

[6] Every bounded sequence has a convergent subsequence.

Proof. It's easy to prove that every bounded sequence has a convergent subsequence. Since, every sequence has a monotonic subsequence and since the sequence is bounded implies subsequence is bounded. And every bounded monotonic sequence is convergent. ■

§2.4.1 Subsequential Limits

Definition 2.4.1

Let (s_n) be a sequence in $\mathbb{R}$. A subsequential limit is any real number or symbol $+\infty$ or $-\infty$ that is the limit of some subsequence of (s_n).

When a sequence has a limit s, then all subsequences have limit s, so {s} is the set of subsequential limits

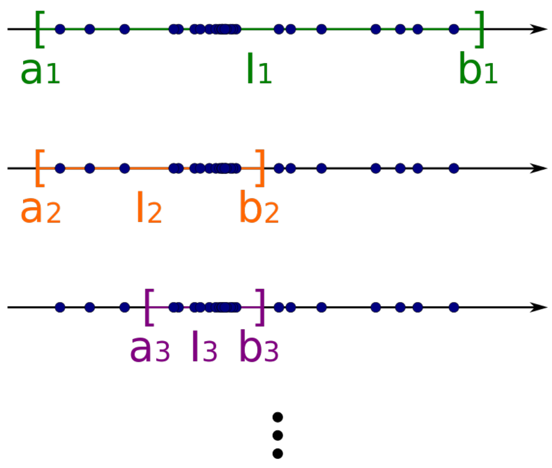


Figure 2.3: Bolzano-Weierstrass Theorem

Theorem 2.4.4

Let (s_n) be any sequence. There exists a monotonic subsequence whose limit is $\limsup s_n$, and there exists a monotonic subsequence whose limit is $\liminf s_n$.

Recall Let (s_n) be any sequence of real numbers, and let S be the set of subsequential limits of (s_n). Recall

$$\liminf s_n = \lim_{N\to\infty} \inf\{s_n \ : \ n > N\} = \inf S$$

and

$$\limsup s_n = \lim_{N\to\infty} \sup\{s_n \ : \ n > N\} = \sup S$$

Chapter 3

Series

§3.1 Sum to infinity?

Definition 3.1.1: Summation Notation

$\sum_n^m a_k = a_n + a_{n+1} + \ldots + a_m$

2.To assign meaning to $\sum_{n=m}^{\infty} a_n$, we consider the sequences $(s_n)_{n=m}^{\infty}$ of partial sums:

$$s_n = a_m + a_{m+1} + \ldots + a_n = \sum_{k=m}^{n} a_k$$

The infinite series $\sum_{n=m}^{\infty} a_n$ an is said to converge provided the sequence (s_n) of partial sums converges to a real number S, in which case we define

$$\sum_{n=m}^{\infty} a_n = S$$

Definition 3.1.2: Cauchy Criterion for Series Convergence

We say a series $\sum a_n$ satisfies the Cauchy criterion if its sequence (s_n) of partial sums is a Cauchy sequence i.e.:

for each $\epsilon > 0$ there exists a number N such that:

$$n \geq m > N \text{ implies } |s_n - s_{m-1}| < \epsilon$$

And, $s_n - s_{m-1} = \sum_n^m a_k$

A series converges iff it satisfies cauchy criterion

Corollary 3.1.1

If $\sum a_n$ converges then $\lim a_n = 0$

§3.2 Convergence Tests for Series

§Comparison Test Let $\sum a_n$ be a series where $a_n \geq 0$ for all n.

1. If $\sum a_n$ converges and $|b_n| \leq a_n$ for all n, then $\sum b_n$ converges.
2. If $\sum a_n = +\infty$ and $b_n \geq a_n$ for all n, then $\sum b_n = +\infty$

Problem 9. *Show that the series $s_n = \sum \frac{1}{n^2}$ converges*

Solution. Observation:

$$\Rightarrow \frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$$

$$\Rightarrow \sum_2^n \frac{1}{n^2} < \sum_2^n \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

Or

$$\Rightarrow 1 + \sum_2^n \frac{1}{n^2} < 1 + \sum_2^n \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

$$\Rightarrow \sum_2^n \frac{1}{n^2} < 2 - \frac{1}{n}$$

As $n \to \infty$ we get:

$$\sum_2^\infty \frac{1}{n^2} < 2$$

Hence, it converges. ■

§ Ratio Test: [7] A series $\sum a_n$ of nonzero terms.The usual form of the test makes use of the limit:

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

(i) if $L < 1$ then the series converges absolutely;

(ii) if $L > 1$ then the series diverges;

(iii) if $L = 1$ or the limit fails to exist, then the test is inconclusive, because there exist both convergent and divergent series that satisfy this case.

It is possible to make the ratio test applicable to certain cases where the limit L fails to exist, if limit superior and limit inferior are used. The test criteria can also be refined so that the test is sometimes conclusive even when L = 1. More specifically, let

- $R = \limsup \left|\frac{a_{n+1}}{a_n}\right|$
- $r = \liminf \left|\frac{a_{n+1}}{a_n}\right|$

Then the ratio test states that:

- if $R < 1$, the series converges absolutely;
- if $r > 1$, the series diverges;
- if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all large n (regardless of the value of r), the series also diverges; this is because $|a_n|$ is nonzero and increasing and hence an does not approach zero;
- the test is otherwise inconclusive

§ **Root Test:** [8] Let $\sum a_n$ be a series and let $\alpha = \limsup |a_n|^{1/n}$. The series $\sum a_n$

(i) converges absolutely if $\alpha < 1$.

(ii) diverges if $\alpha > 1$.

(iii) Otherwise $\alpha = 1$ and the test gives no information

Note that if:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|}$$

converges then it equals α and may be used in the root test instead.

Lemma 3.2.1

Let $|s_n|$ be a sequence of non-zero real numbers,
Then,

$$\liminf\left|\frac{s_{n+1}}{s_n}\right| \leq \liminf\left||s_n|^{\frac{1}{n}}\right| \leq \limsup\left||s_n|^{\frac{1}{n}}\right| \leq \limsup\left|\frac{s_{n+1}}{s_n}\right|$$

Problem 10. *Prove that* $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

Solution. a. Given:

$$\Rightarrow a_n = \frac{x^n}{n!}$$
$$\Rightarrow a_{n+1} = \frac{x^{n+1}}{n+1!}$$
$$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{x}{n+1}$$

As $n \to \infty$ ratio $\frac{a_{n+1}}{a_n} \to 0$. Thus it converges[1] . That means the series $\sum \frac{x^n}{n!}$ converges therefore $\frac{x^n}{n!}$ converges to zero.

b. The series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges. Hence $\frac{x^n}{n!} \to 0$.

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■

Problem 11. *Let* $\limsup |a_n| > 0$. *Then prove that* $\limsup |a_n|^{\frac{1}{n}} \geq 1$.

Proof. Assume that $\limsup |a_n| > 0$ but $\limsup |a_n|^{\frac{1}{n}} < 1$. We also conclude that $\limsup |a_n|^{\frac{1}{n}} < 1$ implies $\sum a_n$ converges absolutely (by Root test).So, $\lim |a_n| = \limsup |a_n| = 0$. Contradiction!. So, $\limsup |a_n|^{\frac{1}{n}} \geq 1$. ■

Theorem 3.2.1

Let $\sum |a_n|$ be a convergent series & let (b_n) be a bounded sequence. Then, $\sum a_n b_n$ is also convergent.

Proof. By triangular inequality we can show that:

$$\left| \sum_{k=m}^{n} a_k b_k \right| \leq \sum_{k=m}^{n} |a_k b_k|$$

Given, $|b_n| \leq M$ (it's bounded), implies:

$$\Rightarrow |a_k||b_k| \leq |a_k| \, M$$

$$\Rightarrow \sum |a_k||b_k| \leq \sum |a_k| \, M$$

Since, $\sum a_n$ converges; therefore by cauchy criterion $\exists \, N \in \mathbb{N}$ such that $\forall \, n \geq m > N$:

$$\Rightarrow \sum_{k=m}^{n} |a_k| < \frac{\epsilon}{M}$$

$$\Rightarrow \sum_{k=m}^{n} M|a_k| < \epsilon$$

Or

$$\Rightarrow \left| \sum_{k=m}^{n} a_k b_k \right| \leq \sum |a_k||b_k| \leq \sum_{k=m}^{n} M|a_k| < \epsilon$$

Hence, $\sum a_n b_n$ is convergent by comparison test. ■

Corollary 3.2.1

Let $a_n \geq 0$ & $\sum a_n$ converges. Then $\sum (a_k)^p$ converges $\forall \, p > 1$

Proof. The above expression can be rewritten as:

$$\sum_{k=m}^{n} |a_k|^p = \sum_{k=m}^{n} |a_k||a_k|^{p-1}$$

Since, $\sum a_n$ converges, therefore $a_n \to 0$. So, sequence a_n is convergent, hence bounded.So, a_k^{p-1} is bounded.

$\therefore$ By previous theorem, we can say $\sum (a_k)^p$ converges. ■

§3.3 Alternating Series Test

Theorem 3.3.1

If $a_1 \geq a_2 \geq \ldots \geq a_n \geq \ldots \geq 0$ and $\lim a_n = 0$, then the alternating series $\sum (-1)^{n+1} a_n$ converges. Moreover, the partial sums $s_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$ satisfy $|s - s_n| \leq a_n$ for all n.

Proof. We need to show that the sequence (s_n) converges. Note that the subsequence (s_{2n}) is increasing because $s_{2n+2} - s_{2n} = -a_{2n+2} + a_{2n+1} \geq 0$. Similarly, the subsequence (s_{2n-1}) is decreasing since $s_{2n+1} - s_{2n-1} = a_{2n+1} - a_{2n} \leq 0$. We claim:

$$s_{2m} \leq s_{2n+1} \quad \text{for all} \quad m, n \in \mathbb{N}$$

First note that $s_{2n} \leq s_{2n+1}$ for all n, because $s_{2n+1} - s_{2n} = a_{2n+1} \geq 0$. If $m \leq n$, then the above equation holds because $s_{2m} \leq s_{2n} \leq s_{2n+1}$. If $m \geq n$, then equation holds because $s_{2n+1} \geq s_{2m+1} \geq s_{2m}$. We see that (s_{2n}) is an increasing subsequence of (s_n) bounded above by each odd partial sum, and (s_{2n+1}) is a decreasing subsequence of (s_n) bounded below by each even partial sum. By Theorem (2.3.1), these subsequences converge, say to s and t. Now

$$t - s = \lim_{n\to\infty} s_{2n+1} - \lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} (s_{2n+1} - s_{2n}) = \lim_{n\to\infty} a_{2n+1} = 0$$

. so s = t, follows that $\lim_n s_n = s$.

To check the last claim, note that $s_{2k} \leq s \leq s_{2k+1}$, so both $s_{2k+1} - s$ and $s - s_{2k}$ are clearly bounded by $s_{2k+1} - s_{2k} = a_{2k+1} \leq a_{2k}$. So, whether n is even or odd, we have $|s - s_n| \leq a_n$. ■

§3.4 Integral Test

Theorem 3.4.1

Consider an integer N and a non-negative function f defined on the unbounded interval $[N, \infty)$, on which it is monotone decreasing. Then the infinite series:

$$\sum_{n}^{\infty} f(n)$$

converges to a real number if and only if the improper integral

$$\int_N^{\infty} f(x)dx$$

is finite. In other words, if the integral diverges, then the series diverges as well.

Problem 12. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence such that* $\liminf |a_n| = 0$. *Prove there is a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that $\sum_{k=1}^{\infty}(a_{n_k})$ converges.*

Proof. We first set $n_0 = 1$ and $c_1 = 1$. By the property of lim inf, there exists $n_1 > n_0 = 1$ such that $|a_{n_1}| < c_1 = 1$. Why does such an n_1 exist? For sake of contradiction, assume there is no $n \in \mathbb{N}$ such that $|a_n| < 1$, then 1 would be a lower bound for $\{a_n : n \in \mathbb{N}\}$. In which case, for all N, 1 would be a lower bound for $\{a_n : n > \mathbb{N}\}$ so that $1 \leq inf\{a_n : n > \mathbb{N}\}$, in which case $1 \leq \lim_{N\to\infty} \inf\{a_n : n > \mathbb{N}\}$, so we would get $1 \leq 0$. A contradiction.

Then we set $c_2 = 1/4$. Again, there exists $n_2 > n_1$ such that $|a_{n_2}| < c_2 = 1/4$. Why does such an n_2 exist? (the same argument can be used here as above)

We can continue this fashion. At the k-th step, we set $c_k = 1/k^2$ and we can then find some $n_k > n_{k-1}$ such that $|a_{n_k}| < c_k = 1/k^2$.

Note that

$$\sum_{k=1}^{\infty} |a_{n_k}| < \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Since the RHS converges, the LHS converges, i.e., $\sum_{k=1}^{\infty} a_{n_k}$ converges absolutely. ■

Problem 13. *Prove if* (a_n) *is a decreasing sequence of real numbers and if* $\sum a_n$ *converges, then* $\lim n \cdot a_n = 0$.

Proof. Since it's given that $\sum a_n$ converges. Therefore by cauchy criteria $\forall \epsilon > 0$ $\exists N \in \mathbb{N}$ such that $\forall n > m+1 > N$: (we'll be using the fact that for $n > m$ we have $a_n < a_m$)

$$\sum_{m+1}^{n} a_k < \epsilon$$

we didn't use a mod because observe that all terms are positive

$$(n-m)a_n \leq a_{m+1} + a_{m+2} \cdots + a_n < \epsilon$$

Hence, $\lim_{n\to\infty}(n-m)a_n = 0$. Further, we see that :

$$\lim_{n\to\infty} na_n = \lim_{n\to\infty}(n-m)a_n + \lim_{n\to\infty} ma_n = 0$$

■

Problem 14. *Show that* $\sum \frac{1}{n^p}$ *converges iff* $p > 1$

Proof.

1. In particular, for $p \leq 1$, we can write $\sum \frac{1}{n^p} = +\infty$. For $p = 2$ we have already proved it in Problem 9.

   For $p > 2$, we can prove it by the comparison test as :

   $$\left\{ \frac{1}{n^p} < \frac{1}{n^2} \right\}, \quad \forall \quad p > 0$$

   Since, $\frac{1}{n^2}$ converges. Then $\frac{1}{n^p}$ converges by comparison test.

   Or

   The above proposition can be used to prove the result for $p > 2$

   For $1 < p < 2$

   We have,

   $$\underbrace{\sum_{k=1}^{n} \frac{1}{k^p}}_{\substack{\text{sum of rectangles going} \\ \text{from } 1 \to \text{n}}} \leq \underbrace{1 + \int_1^n \frac{1}{x^p} dx}_{\text{area under the curve}}$$

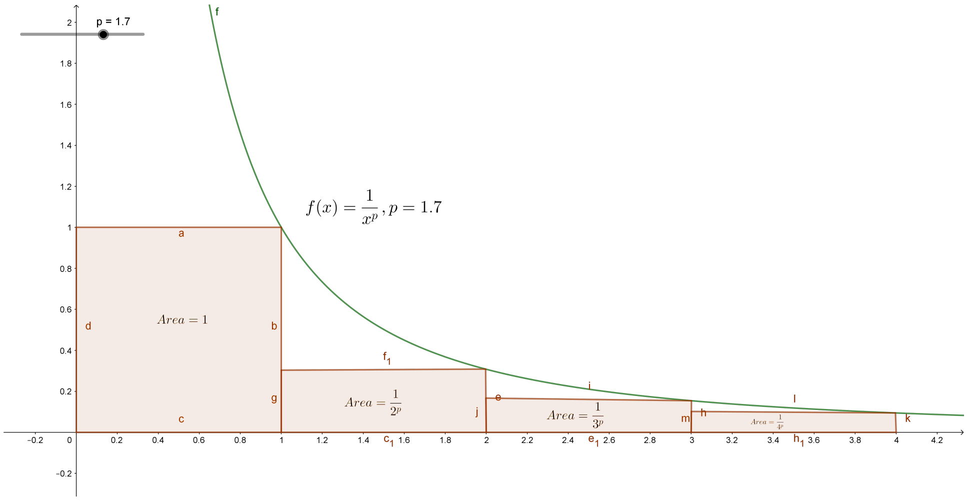
2. 

Figure 3.1: Geometric intution for $\sum \frac{1}{n^p}$

$$\Rightarrow \sum_{k=1}^{n} \frac{1}{k^p} \leq 1 + \left[\frac{x^{1-p}}{1-p}\right]_1^n$$

$$\Rightarrow \sum_{k=1}^{n} \frac{1}{k^p} \leq 1 + \left[\frac{n^{1-p}}{1-p} - \frac{1}{1-p}\right]$$

$$\Rightarrow \sum_{k=1}^{n} \frac{1}{k^p} \leq \left[\frac{p - n^{1-p}}{p-1}\right]$$

As $n \to \infty$ we get($\because 1 - p < 0$),

$$\Rightarrow \sum_{k=1}^{n} \frac{1}{k^p} \leq \left[\frac{p}{p-1}\right]$$

$\blacksquare$

Chapter 4

Continuity

§4.1 Continuity and Functions

Definition 4.1.1

A function f whose domain is defined over $\mathbb{R}$ is said to be continuous at a point $x_0 \in dom(f)$ iff:

for each $\epsilon > 0$ there exist a $\delta > 0$ such that:

$$x \in dom(f) \textit{ and } |x - x_0| < \delta \textit{ imply } \big|f(x) - f(x_0)\big| < \epsilon$$

.

Or

Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. The function f is continuous at $(x_0) \in dom(f)$ if, for every sequence $(x_n) \in dom(f)$ converging to x_0, we have $lim_n f(x_n) = f(x_0)$ i.e.:

if for every sequence in domain if $x_n \to x_0$ we have $f(x_n) \to f(x_0)$

Theorem 4.1.1

Let f be a real-valued function with $dom(f) \subseteq \mathbb{R}$. If f is continuous at x_0 in $dom(f)$, then $|f|$ and kf, $k \in \mathbb{R}$, are continuous at x_0

§4.1.1 Properties of Continuous Functions

§ Bounded Function A real-valued function f is said to be bounded if $\{f(x) : x \in dom(f)\}$ is a bounded set, i.e., if there exists a real number M such that $|f(x)| \leq M$ for all $x \in dom(f)$.

Theorem 4.1.2

Let f be a continuous real-valued function on a closed interval $[a, b]$. Then f is a bounded function. Moreover, f assumes its maximum and minimum values on $[a, b]$; that is, there exist $x_0, y_0 \in [a, b]$ such that $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in [a, b]$.

Theorem 4.1.3: Intermediate value theorem

If f is a continuous real-valued function on an interval I, then f has the intermediate value property on I: Whenever $a, b \in I$, $a < b$ and y lies between $f(a)$ and $f(b)$ i.e. $[f(a) < y < f(b)$ *or* $f(b) < y < f(a)]$, there exists at least one $x \in (a, b)$ such that $f(x) = y$.

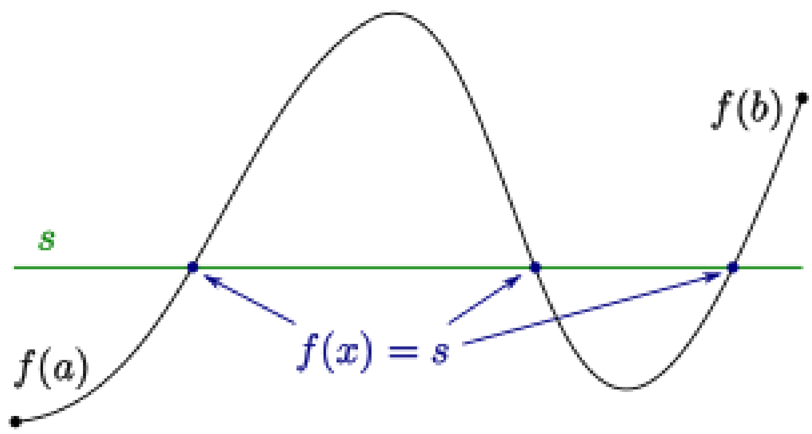


Figure 4.1: Intermediate value theorem

Problem 15. *Prove that* $\lim_{x \to a} |x + 2| = |a + 2|$

Proof. Simply take, $\delta = \epsilon$. We have $|x - a| < \delta$.

$$\big||x + 2| - |a + 2|\big| \leq |x - a| < \delta = \epsilon \quad \text{(By triangular inequality)}$$

■

Problem 16. *Prove that* $\lim_{x \to a} (4 + x - 3x^2) = (4 + a - 3a^2)$

Proof. We need to prove that

$$\left|4 + x - 3x^2 - (4 + a - 3a^2)\right| = \left|(x - a) - 3(x - a)(x + a)\right| < \epsilon$$

Or

$$|x-a|\big|1-3(x+a)\big|<\epsilon$$

Since, we are trying to find a δ we can apply the triangular inequality on the above expression and we get.

$$|x-a|\big|1-3(x+a)\big| \leq |x-a|(1+3|x+a|)<\epsilon$$

We need to find a upper bound for the above expression $(|1+3\big|(x+a)\big|)$ which is not dependet on x.

If we choose $|x-a|<\delta<\frac{\epsilon}{|1+3|x+a||}$ then it would imply $|x-a|\big|1+3(x+a)\big|<\epsilon$. But when we have one $\delta>0$ that works, any smaller value will also work. By choosing $\delta<1$ we would have:

$$|x-a|<1 \quad \text{when ever} \quad |x-a|<\delta$$

$$\Rightarrow ||x|-|a|| \leq |x-a|<1$$

Or

$$\Rightarrow |x|<1+|a|$$
$$\Rightarrow |x+a| \leq |x|+|a|<|2a|+1$$
$$\Rightarrow [1+3|x+a|\ ]<[1+3(1+2|a|)\ ]$$

Since, we need both the above assumptions i.e. $|x-a|<\delta$ & $|x-a|<1$ to be satisfied.So, take :

$$\delta=\min\left(1, \frac{\epsilon}{1+3(1+2|a|)}\right)$$

And, whole discussion above proves that it would work.

■

Problem 17. *Is* $\lim \frac{4x+1}{3x-4}$ *is continuous for* $x \neq 4/3$*?*

Problem 18. *Let f and g be continuous functions on $[a,b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove $f(x_0)=g(x_0)$ for at least one x_0 in $[a,b]$*

Proof. Define $h(x)=f(x)-g(x)$. It's given that $f(a)-g(a) \geq 0$ and $f(b)-g(b) \leq 0$. So, by intermediate value theorem there exists a $x_0 \in [a,b]$. Such that $h(x_0)=0$. ■

Problem 19. *Suppose f is continuous on $[0,2]$ and $f(0)=f(2)$. Prove there exist $x,y \in [0,2]$ such that $|y-x|=1$ and $f(x)=f(y)$.*

Proof. Define $g(x)=f(x+1)-f(x)$. Let $|x-y|=1$ implies either $x=y+1$ or $y=x+1$. Take anyone of them, it won't matter as we can exchange y with x for another case.

Also, $g(0)=f(1)-f(0)$ and $g(1)=f(2)-f(1)$. Adding both the equations $g(0)+g(1)=0$, which implies that one of them is negative of other i.e. if any one of the value is + ve the other is - ve. So, there exists a value of $x \in [0,1]$ such that $g(x)=0$ or $f(x+1)=f(x) \equiv f(y)=f(x)$. ■

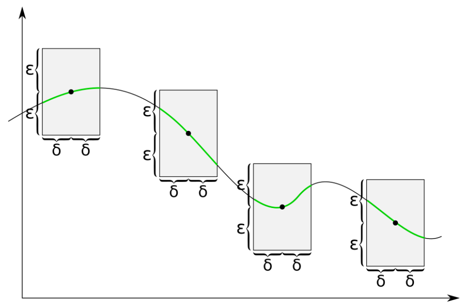
§4.2 Uniform Continuity

Definition 4.2.1: Uniform Continuity

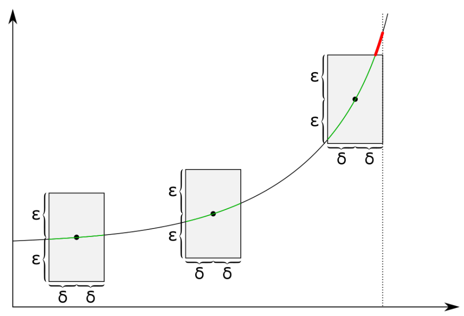
Let f be a real-valued function defined on a set $S \subseteq \mathbb{R}$. Then f is uniformly continuous on S if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$x, y \in S \text{ and } |x - y| < \delta \text{ imply } \left|f(x) - f(y)\right| < \epsilon$$

We will say f is uniformly continuous if f is uniformly continuous on dom(f).



(a) For uniformly continuous functions, there is for each $\epsilon > 0$ a $\delta > 0$ such that when we draw a rectangle around each point of the graph with width 2δ and height 2ϵ, the graph lies completely inside the rectangle.



(b) For functions that are not uniformly continuous, there is an $\epsilon > 0$ such that regardless of the $\delta > 0$ here are always points on the graph, when we draw a $2\epsilon - 2\delta$ rectangle around it, there are values directly above or below the rectangle.

Figure 4.2: Uniform Continuity

If your function happens to satisfy $0 < |f'(x)| < M$ for every x, then something like $\delta = \epsilon/M$ will probably work to show uniform continuity.

Theorem 4.2.1

If f is continuous on a closed interval $[a, b]$, then f is uniformly continuous on $[a, b]$.

Theorem 4.2.2

If f is uniformly continuous on a set S and (s_n) is a Cauchy sequence in S, then $(f(s_n))$ is a Cauchy sequence.

Problem 20. *Prove that if f is uniformly continuous on a bounded set S, then f is a bounded function on S.*

Proof. Since, uniformly continuous functions are cauchy on a closed interval (and if the interval is not closed then we can close it! We have that power.; and cauchy sequences are bounded. ■

Definition 4.2.2

Let f be a function defined domain of f. Another function $\tilde{\mathrm{f}}$ is called an extension of f if.

$$\left\{ \begin{array}{c} \mathrm{dom(f)} = \mathrm{dom}(\tilde{\mathrm{f}}) \\ \mathrm{f(x)} = \tilde{\mathrm{f}}\mathrm{(x)} \end{array} \right\} \forall x \in \mathrm{dom(f)}$$

Let f be defined on (a, b), if f is uniformly continuous. Then, f can be extended to $\tilde{\mathrm{f}}$ on $[a, b]$.

Theorem 4.2.3

A real-valued function f on (a, b) is uniformly continuous on (a, b) if and only if it can be extended to a continuous function $\tilde{f}$ on $[a, b]$.

Theorem 4.2.4

Let f be a continuous function on an interval I [I may be bounded or unbounded]. Let I^o be the interval obtained by removing from I any endpoints that happen to be in I. If f is differentiable on I^o and if f' is bounded on I^o, then f is uniformly continuous on I.

Problem 21.

1. *Let f be a continuous real-valued function with domain (a, b). Show that if $f(r) = 0$ for each rational number r in (a, b), then $f(x) = 0$ for all $x \in (a, b)$*
2. *Let f and g be continuous real-valued functions on (a, b) such that $f(r) = g(r)$ for each rational number r in (a, b). Prove $f(x) = g(x)$ for all $x \in (a, b)$*

Proof. 1. Suppose, towards a contradiction, that there is an $x \in [a,b]$ with . As f is continuous, there is for every $\epsilon = |f(x)|/2$ some $\delta > 0$ such that for all $x' \in [a,b]$ with $|x' - x| < \delta$ it follows that $|f(x) - f(x')| < \epsilon$. As , there is however a rational $x' \in [a,b]$ with $|x - x'| < \delta$. But now $|f(x) - \underbrace{f(x')}_{0}| < |f(x)| < |f(x)|/2$. Contradiction!

Or

Let $x \in [a,b]$, and let q_n be a sequence of rational numbers, such that $q_n \to x$. By continuity of f, we have:

$$f(x) = f(\lim_{n\to\infty} q_n) = \lim_{n\to\infty} f(q_n) = 0$$

Using the above result we can define a function $f(x) - g(x)$, and $f(x) = g(x)$ or $f(x) - g(x) = 0$ for all rationals.

■

Problem 22. *Let*

$$f(x) = \begin{cases} 0; & \textit{for } x \textit{ irratonal} \\ \frac{1}{q}; & \textit{for } x = \frac{p}{q} \\ 1, & \textit{for } x = 0 \end{cases}$$

Show f is continuous at each point of $\mathbb{R} \sim \mathbb{Q}$ *and discontinuous at each point of* $\mathbb{Q}$.

Proof. It's easy to proof for x $\in \mathbb{Q}$. Take a sequence (x_n) of irrationals such that $x_n \to p/q$. But, $f(x_n) = 0 \neq f(p/q) = 1/q$. Also, observe that the function is periodic with period 1, i.e. $f(x) = f(x+1)$.

For x in set of irrational numbers. For a sequence of irrationals it's obvious that the difference b/w the function values will be 0 always. For a sequence of rationals we can think of any sequence of rationals coverging to x, then the denominator value will approach infinity(it's not a formal proof). An example is shown below shown in the footnote below. ■

Problem 23. *Let*

$$f(x) = \begin{cases} 0; & \textit{for } x \textit{ irratonal} \\ 1; & \textit{for } x \textit{ rational} \end{cases}$$

Show f is discontinuous at each point of $\mathbb{R}$.

Proof. We begin by considering a sequence of irrational numbers x_n converging to a rational x_0. $x_n = x_0 + \frac{\lambda}{n}$ where $\lambda \in \mathbb{R} \sim \mathbb{Q}$.Since, every element of x_n is irrational implies $f(x_n) = 0 \neq f(x_0) = 1$. So, there exist a sequence in the domain such that $x_n \to x_0$ but $f(x_n) \not\to f(x_0)$ for rational x_0

Similarly, using the density property of rational numbers [1]there exist a sequence of rational numbers (x_n) in $\mathbb{R}$ such that $x_n \to x_0$ for x_0 irrational.But $f(x_n) = 1 \neq f(x_0) = 0$. Hence , it is not continuous. ■

[1]you need a sequence of rationals converging to the irrational x. In theory, we already know one: consider the decimal expansion of x. When x is irrational, the sequence is necessarily infinite, doesn't eventually repeat itself forever. Suppose

$$x = m + 0.d_1 d_2 \ldots$$

Problem 24. *Let f be a continuous function on $[0, \infty)$. Prove that if f is uniformly continuous on $[k, \infty)$ for some k, then f is uniformly continuous on $[0, \infty)$.*

Problem 25. *Let f be a continuous function on $[a, b]$. Show that the function f^* defined as $f^*(x) = \sup\{f(y) : a \leq y \leq x\}$, for $x \in [a, b]$, is an increasing continuous function on $[a, b]$.*

Proof. Take $x_2 > x_1$. Let $S_1 = [a, x_1]$ & $S_2 = [a, x_2]$. Observe $S_1 \subset S_2$.By definition $f^*(x_1) = \sup\{f(y) : a \leq y \leq x_1\}$ & $f^*(x_2) = \sup\{f(y) : a \leq y \leq x_2\}$. So, $f(x_1) \geq f(x) \forall x \in S_1$ and $f(x_2) \geq f(x) \forall x \in S_2$. Since, $S_1 \subset S_2$ so, $x_1 \in S_2$ implies $f(x_2) \geq f(x_1)$. ■

§4.3 Limits of Functions

In this section we'll formalize the notion of limit of a function and this will help us for a careful study of derivatives

Definition 4.3.1

Let $S \subseteq \mathbb{R}$ and $S \neq \varnothing$, let a be a real number or symbol $-\infty$ or ∞ that is a limit of a sequence in S. And let L be a real number or symbol $-\infty$ or ∞. We write $\lim_{x \to a} f(x) = L$ if:

$$\text{f is a function defined on S}$$

and

$$\text{for every sequence } (\mathrm{x_n}) \text{ in S converging to a, we have } \lim_{n \to \infty} f(x_n) = L$$

The expression "$\lim_{x \to a^S} f(x)$" is read as "limit, as x tends to a along S,of f(x),"

Notations

$$S = (-\infty, b) : \lim_{x \to a} f(x) = \lim_{x \to a^-} f(x)$$

and

$$S = (a, \infty) : \lim_{x \to a} f(x) = \lim_{x \to a^+} f(x)$$

Question: Let a be a real number & $S = (a, b) \subseteq \mathbb{R}$. Does $\lim_{x \to a^S} f(x)$ depends on S. If we take a different set $T = (a, b_1) \subseteq \mathbb{R}$ then what's the relation b/w $\lim_{x \to a^S} f(x)$& $\lim_{x \to a^T} f(x)$

---

where m is an integer. Let $x_n = m + 0.d_1 \ldots d_n$ In other words, x_n is the decimal representation of x cut off at the $\mathrm{n_{th}}$ digit after the decimal point $(x_0 = m)$. Then every x_n is rational, and:

$$\lim_{n \to \infty} x_n = x$$

Answer: Since, as the $t_n \in T$ converge to a, then after some iterations the elements of sequence t_n will be the elements of set S (Assuming $S \subseteq T$).then:

$$\lim_{x \to a^S} f(x) = \lim_{x \to a^T} f(x)$$

Theorem 4.3.1

Let f be a function for which the limit $L = \lim_{x \to a^S} f(x)$ exists and is finite. If g is a function defined on $\{f(x) : x \in S\} \cup \{L\}$ that is continuous at L, then $\lim_{x \to a^S} g \circ f(x)$ exists and equals g(L).

Theorem 4.3.2

Let f be a function defined on a subset S of R, let a be a real number that is the limit of some sequence in S, and let L be a real number. then $\lim_{x \to a^S} f(x) = L$ if and only if

for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$x \in S \text{ and } |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon. \tag{4.1}$$

Proof. Assuming the statement and proving (3) is trivial so I leave it.

Let $\lim_{n \to \in\infty} f(x_n) = L$:

Now assume $\lim_{n \to \infty} x_n = a$, but (3) fails. So, there exist a $\epsilon > 0$ such that for all $\delta > 0$ $|f(x_n) - f(a)| \geq \epsilon$. Take $\delta = 1/n$.So, it implies for all $n \in \mathbb{N}$ there exist x_n in S such that $|x_n - a| < 1/n$ while $|f(x_n) - f(x_0)| \geq \epsilon$. Hence aur assumption was wrong and there exist a sequence (x_n) which converges to a , but $f(x_n)$ doesn't converge to $f(a)$ ■

Corollary 4.3.1

Let f be a function defined on $J \backslash \{a\}$ for some open interval J containing a, and let L be a real number. Then $\lim_{x \to a} f(x) = L$ iff:

for each $\epsilon > 0$ there exists $\delta > 0$ such that
$|x - a| < \delta$ implies $|f(x) - L| < \epsilon$.

Chapter 5

Sequence and Series of Functions

§5.1 Power Series

Definition 5.1.1: Power Series

$\sum_0^\infty a_n x^n$ is called a Power Series.

Definition 5.1.2: Radius of Convergence

Let $\beta = \limsup |a_n|^{\frac{1}{n}}$.Then **Radius of convergence** is defined as:

$$R = \frac{1}{\beta}$$

Theorem 5.1.1

1. $\sum_0^\infty a_n x^n$ converges for $|x| < R$.
2. $\sum_0^\infty a_n x^n$ diverges for $|x| > R$.

Proof. Take $t \in \mathbb{R}$. Take $r_t = \limsup |a_n t^n|^{\frac{1}{n}}$.Then

$$r_t = \limsup |a_n t^n|^{\frac{1}{n}} = |t| \limsup |a_n|^{\frac{1}{n}} = |t|\beta$$

Case 1. Let $0 < R < \infty$

$$r_t = \beta t = \frac{|t|}{R} = \begin{cases} |t| < R \Rightarrow & \overbrace{r_t < 1}^{\text{converges by root test}} \\ |t| > R \Rightarrow & \underbrace{r_t > 1}_{\text{diverges by root test}} \end{cases}$$

Case 2. Let $R = \infty$
Here $\beta = 0$ implies $r_t = 0 < 1$.So, converges for all x by root test.

Case 3. Let $R < 0$
$\Rightarrow \beta = \infty$ implies $r_t = \infty$; for all $|t| \neq 0$. Therefore Diverges. ■

Example: $\sum x^n$. Here $a_n = 1$.

$$\therefore \lim \left|\frac{a_{n+1}}{a_n}\right| = 1 \quad \therefore \beta = 1, R = 1$$

. Therefore the series converges for $|x| < 1$. A strict inequality!.

Example: Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

The radius of convergence for the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(y)^n$ is $R = 1$, so it converges for $|y| < 1$ or $x \in (0, 2)$ at $x = 0$ and $y = -1$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(-1)^n = -\sum \frac{1}{n}$$

therefore it diverges to $-\infty$. at x = 2

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(1)^n = \sum \frac{(-1)^{n+1}}{n} = -\ln 2$$

Hence,it converges & the interval of convergence is $(0, 2]$.

Problem 26. *Consider a power series* $\sum a_n x_n$ *with radius of convergence R.Prove that if all the coefficients an are integers and if infinitely many of them are nonzero, then* $R \leq 1$.

Proof. Given $|a_n| \geq 1$ ($\because$ a_n are integers). So, $s_n = \sup\{|a_k| : k \geq n\} \geq 1$. Further $|s_n|^{\frac{1}{n}} \geq 1$ implies $\lim |s_n|^{\frac{1}{n}} \geq 1$. Or $R \leq 1$ using 5.1.2. ■

Problem 27. *Prove that if* $\limsup |a_n| > 0$*, then* $\limsup |a_n|^{\frac{1}{n}} \geq 1$.

Proof. Let $\limsup |a_n| > 0$ (notice the strict inequality!). But for sake of contradiction assume $\limsup |a_n|^{\frac{1}{n}} < 1$. Then $\sum a_n$ converges, which implies $\lim a_n = 0$ (See 3.1.1). A contradiction! ■

§5.2 Uniform Convergence

Definition 5.2.1: Pointwise Convergence

Let (f_n) be a sequence of real-valued functions defined on a set $S \subseteq \mathbb{R}$. The sequence (f_n) converges pointwise [i.e., at each point] to a function f defined on S if

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for all} \quad x \in S$$

We often write $\lim f_n = f$ pointwise [*on S*] or $f_n \to f$ pointwise [on S].

Now observe $f_n \to f$ pointwise on S means exactly the following:
for each $\epsilon > 0$ and x in S there exists N such that

$$|f_n(x) - f(x)| < \epsilon \quad \text{for} \quad n > N$$

Note the value of N depends on both ϵ and x in S.

Example Let $f_n(x) = x^n$ for $x \in [0, 1]$. Then $f_n \to f$ pointwise on $[0, 1]$ where $f(x) = 0$ for $x \in [0, 1)$ and $f(1) = 1$. Or we can write:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{else} \end{cases}$$

Consider,

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{else} \end{cases}$$

Then we can write $\lim_{n\to\infty} f_n(x) = f(x)$.

Definition 5.2.2: Uniform Convergence

Let (f_n) be a sequence of real-valued functions defined on a set $S \subseteq \mathbb{R}$. The sequence (f_n) converges uniformly on S to a function f defined on S if
for each $\epsilon > 0$ there exists a number N such that

$$|f_n(x) - f(x)| < \epsilon \quad \text{for all} \quad x \in S \quad \textit{and} \quad n > N$$

. We write $\lim f_n = f$ uniformly on S or $f_n \to f$ uiniformly on S.

Example: Define,

$$f_n(x) = x^n \quad for \quad x \in [0, 1]$$

$$f(x) = \begin{cases} 0, x \neq 0 \\ 1, x = 0 \end{cases}$$

then $\epsilon = 1/2$,then we consider for all $x \in [0, 1]$ and all $n > N$.

$$|f_n(x) - f(x)| < \epsilon = 1/2$$

$|x| \leq 1$ & $n > N$

$$\Rightarrow |f_n - f| < 1/2$$

at $x = 0$ it's not possible as $|0 - 1| < 1/2$ is not true. So, not uniformly converging.

Example Let $f(x) = \frac{1}{n}sin(nx)$ $\forall x \in \mathbb{R}$

for any x $\lim_{n\to\infty} f_n(x) = 0$ pointwise on R. Define $f(x) = 0$ then $f_n \to f$. In fact $f_n \to f$ uniformly on R. Also, let $N = 1/\epsilon$ .Then for $n > N$ and all $x \in \mathbb{R}$ we have

$$|f_n(x) - f(x)| = |f_n(x) - 0| = \left|\frac{1}{n}sin(nx)\right| \le \frac{1}{n} < \frac{1}{N} = \epsilon$$

Since, N is independent of x.

Theorem 5.2.1

Uniform limit of continuous function is continuous.More precisely, let (f_n) be a sequence of functions on a set $S \subseteq \mathbb{R}$, suppose $f_n \to f$ uniformly on S, and suppose $S = dom(f)$. If each f_n is continuous at x_0 in S, then f is continuous at x_0. [So if each f_n is continuous on S, then f is continuous on S.]

Remark Remember $f_n \to f$ on S uniformly iff,

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in S\} = 0$$

We can also consider $\sum_{k=0}^{\infty} g_k(x)$, where g_k is a function of $S \subset \mathbb{R}$.

Theorem 5.2.2

If g_k is continuous $\forall k$ and if $\sum g_k$ converges uniformly, then it converges to a continuous function on S.

Proof. Let $f_n = \sum_{k=0}^{n} g_k(x)$ is continuous on S and $f_n(x) = \sum_{k=0}^{\infty} g_k(x)$ uniformly.

$\therefore f(x) = \sum_{k=0}^{\infty} g_k(x)$ is continuous. ■

Corollary 5.2.1

$f(x) = \sum a_n x^n$ is a continuous function if the convergence is uniform.

Problem 28. *Prove that if (f_n) is a sequence of uniformly continuous functions on an interval (a, b), and if $f_n \to f$ uniformly on (a, b), then f is also uniformly continuous on (a, b).*

Proof. We need to show that $\forall \delta > 0$ $\exists \epsilon > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

Using triangualar inequality we can show that:

$$|f(x) - f(y)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$

Using the $\epsilon/3$ argument we can show that, there exists a N such that $n > N_1$ implies $|f_n(x) - f_n(y)| < \epsilon/3$ (it's uniform cont.) and $|x - y| < \delta$.And we can apply definition of uniform convergence to show $|f_n(y) - f(y)| < \epsilon/3$ by choosing some N_2 for x and then some N_2 for y.So we have for $n > N = max(N_1, N_2, N_3)$.

$$|f(x) - f(y)| < 3 \cdot \frac{\epsilon}{3} = \epsilon$$

■

Chapter 6

Differentiation

§6.1 Why differentiation? Better ask Newton.

Definition 6.1.1

f is a real-valued function on an open interval I Let $a \in I$. Then f is differentiable at "a" if the limit :

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists & is finite. We also sayy f has a derivative at a.

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Remark '$f'(x)$ itself is a function with domain $f \subseteq dom f$.

Theorem 6.1.1

If f be diff. at $x = a$. Then f is continuous at a.

Proof. $f(x) = (x - a)\frac{f(x)-f(a)}{x-a} + f(a)$

$$\therefore \lim_{x \to a} f(x) = \lim_{x \to a}(x - a) \lim_{x \to a} \frac{f(x) - f(a)}{x - a} + \lim_{x \to a} f(a) = 0 \cdot f'(a) + f(a) \tag{6.1}$$

(6.2)

■

Theorem 6.1.2: Properties

If f, g are diff. at $x = a$ and $c \in \mathbb{R}$ then

1. $(cf)'(a) = cf'(a)$
2. $(f + g)'(a) = f'(a) + g'(a)$

3. $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$
4. $(\frac{f}{g})'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$

Theorem 6.1.3: Chain Rule

f be differentiable at $x = a$ and g be differentiable at $f(a)$. Then $(g \circ f)(x)$ is diff at x = a. And $(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$

§6.2 Mean Value Theorem

Theorem 6.2.1

If f is defined on an open interval containing x_0, if f assumes its maximum or minimum at x_0, and if f is differentiable at x_0, then $f'(x_0) = 0$.

Theorem 6.2.2: Rolle's Theorem

Let f be a continuous function on $[a, b]$ that is differentiable on (a, b) and satisfies $f(a) = f(b)$. There exists [at least one] x in (a, b) such that $f'(x) = 0$.

Theorem 6.2.3: Mean Value Theorem

Let f be a continuous function on $[a, b]$ that is differentiable on (a, b). Then there exists [at least one] x in (a, b) such that:

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

Corollary 6.2.1

Let f be a differentiable function on (a, b) such that $f'(x) = 0$ for all $x \in (a, b)$. Then f is a constant function on (a, b).

Corollary 6.2.2

Let f and g be differentiable functions on (a, b) such that $f' = g'$ on (a, b). Then there exists a constant c such that $f(x) = g(x) + c$ for all $x \in (a, b)$.

Corollary 6.2.3

Let f be a differentiable function on interval (a, b), then:

(i.) f is strictly increasing if $f'(x) > 0$ for all $x \in (a, b)$;

(ii.) f is strictly decreasing if $f'(x) < 0$ for all $x \in (a, b)$;

(iii.) f is increasing if $f'(x) \geq 0$ for all $x \in (a, b)$;

(iv.) f is decreasing if $f'(x) \leq 0$ for all $x \in (a, b)$;

Theorem 6.2.4: Intermediate Value Theorem for Derivatives.

Let f be a differentiable function on (a, b). If $a < x_1 < x_2 < b$, and if c lies between $f'(x_1)$ and $f'(x_2)$, there exists [at least one] x in (x_1, x_2) such that $f'(x) = c$.

Problem 29. *Let f be differentiable on R with $a = \sup\{|f'(x)| : x \in \mathbb{R}\} < 1$. Select $x_0 \in \mathbb{R}$ and define $x_n = f(x_{n-1})$ for $n \geq 1$. Thus $x_1 = f(x_0), x_2 = f(x_1)$, etc. Prove (x_n) is a convergent sequence.*

Proof. Using the mean value theorem we can see f is a contraction, because

$$|f(x) - f(y)| = |f'(c)||x - y| \leq a|x - y|$$

With $a < 1$.

Now to see it is a convergent / cauchy sequence, select x_0 and note $x_n = f(x_{n-1})$, what can you say of

$$|x_n - x_m|$$

for arbitrary m, n?

HINT: Use triangle inequality and $|x_n - x_{n+1}| \leq a^n|x_0 - x_1|$. For alternate proof refer to Problem 6

Suppose $m > n$ then:

$$|x_n - x_m| \leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} + x_m|$$

$$\leq \sum_{k=n}^{m} a^k|x_0 - x_1| \leq \sum_{k=n}^{\infty} a^k|x_0 - x_1| < +\infty$$

Now, because this last series converges $(a < 1)$, by the Cauchy criterion, given $\varepsilon > 0$, there is some N such that

$$\sum_{k=N}^{\infty} a^k|x_0 - x_1| < \varepsilon$$

If we pick $n, m \geq N$ we're done, now we know x_n is a Cauchy sequence in $\mathbb{R}$, and thus has a limit x, now

$$x = \lim x_n = \lim f(x_{n-1}) = f(\lim x_{n-1}) = f(x)$$

hence x is a fixed point. To see its uniquenes suppose x_1, x_2 are two fixed points, then:

$$|x_1 - x_2| = |f(x_1) - f(x_2)| \leq a|x_1 - x_2|$$

With $a < 1$ this is only true if $x_1 = x_2$. ∎

§6.3 Taylor Theorem

Definition 6.3.1: Taylors Series

Let f be a function defined on some open interval containing c. If $f^k(c)$ exists $\forall k$, then the series:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

is called as **Taylor Series** of function $f(x)$ about c. For $n \geq 1$; remainder $R_n(x)$ is defined as:

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

The remainder is important because, for any x;

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \quad \text{if and only if} \quad \lim_{n\to\infty} R_n(x) = 0$$

Theorem 6.3.1

Let f be defined on (a,b) where $a < c < b$; here we allow $a = -\infty$ or $b = \infty$. Suppose the n^{th} derivative f^n exists on (a,b). Then for each $x \neq c$ in (a,b) there is some y between c and x such that

$$R_n(x) = \frac{f^n(y)}{n!}(x-c)^n$$

Corollary 6.3.1

Let f be defined on interval (a,b) and $a < c < b$. If f^k exists $\forall k$, & there exists a single constant C such that $\left|f^k(c)\right| \leq C$, then

$$\lim_{n\to\infty} R_n(x) = 0 \quad \forall\ x \in (a,b)$$

Chapter 7

Die Theorie der Integration

§7.1 Integrals

Definition 7.1.1: The Darboux Integral

Let f be a bounded function on a closed interval $[a, b]$.1 For $S \subseteq [a, b]$, we adopt the notation:

- $M(f, S) = \sup\{f(x) : x \in S\}$
- $m(f, S) = \inf\{f(x) : x \in S\}$.

A partition of $[a, b]$ is any finite ordered subset P having the form

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}.$$

The upper Darboux sum $U(f, P)$ of f with respect to P is the sum:

$$U(f, P) = \sum \overbrace{M(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})}^{\text{Area of rect. above the curve}}$$

and the lower Darboux sum $L(f, P)$ is:

$$L(f, P) = \sum \underbrace{m(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})}_{\text{Area of rect. below the curve}}$$

Note

$$U(f, P) \leq M(f, [a, b])(b - a);$$

and

$$L(f, P) \geq m(f, [a, b])(b - a);$$

The upper Darboux integral $U(f)$ of f over $[a, b]$ is defined by

$$U(f) = \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}$$

and the lower Darboux integral is

$$L(f) = \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}$$

We say $f(x)$ is integrable if on $[a, b]$ if $L(f) = U(f)$.

Observe: $L(f,\{a,b\}) \leq L(f,P) \leq U(f,P) \leq U(f,\{a,b\})$.

Lemma 7.1.1

Let f be a bounded function on $[a,b]$. If P and Q are partitions of $[a,b]$ and $P \subseteq Q$, then
$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$$

Lemma 7.1.2

If f is a bounded function on $[a,b]$, and if P and Q are partitions of $[a,b]$, then $L(f,P) \leq U(f,Q)$

Lemma 7.1.3

If f is a bounded function on $[a,b]$, then $L(f) \leq U(f)$.

Theorem 7.1.1

A bounded function f on $[a,b]$ is integrable if and only if for each $\epsilon > 0$ there exists a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \epsilon$$

Definition 7.1.2: Mesh

The *mesh* of a partition P is the maximum length of the subintervals comprising P. Thus if:

$$P = \{a = t_0 < t_1 < t_2 \cdots < t_n = b\},$$

then
$$\mathrm{mesh}(P) = \max\{t_k - t_{k-1} : k = 1,2,3,\ldots,n\}$$

Definition 7.1.3: Rienmann Sum

Let $f : [a,b] \to \mathbb{R}$ be a function defined on a closed interval $[a,b]$ of the real numbers, $\mathbb{R}$, and
$$P = \{a = t_0 < t_1 < t_2 \cdots t_n = b\}$$
be a partition of $[a,b]$. A riemann sum of f associated with the partition P is a sum of the form
$$\sum_{k=1}^{n} f(x_k)(t_k - t_{k-1})$$
where $x_k \in [t_{k-1}, t_k]$ for $k = 1,2,3,\ldots n$. The choice of x_k is arbitrary, so there are infinitely many Riemann sums associated with a single function and

partition.

Definition 7.1.4: Riemann Integral

The function f is Riemann integrable on $[a, b]$ if there exists a number r such that for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$|S - r| < \delta$$

for every Rienmann sum of f associated with the partition P having mesh$(P) < \delta$. The number r is the Riemann integral of f on $[a, b]$ and will be provisionally written as $\mathcal{R} \int_a^b f$.

Theorem 7.1.2

A bounded function f on $[a, b]$ is integrable if and only if for each $\epsilon > 0$ there exists a $\delta > 0$ such that $mesh(P) < \delta$ implies

$$U(f, P) - L(f, P) < \epsilon$$

for all partitions P of $[a, b]$.

§7.1.1 Fundamental Theorem of Calculus

Theorem 7.1.3

If g is a continuous function on $[a, b]$ that is differentiable on (a, b), and if g' is integrable on $[a, b]$, then

$$\int_a^b g' = g(b) - g(a)$$

Theorem 7.1.4

Let f be an integrable function on $[a, b]$. For x in [a, b], let

$$F(x) = \int_a^x f(t)dt.$$

Then F is continuous on $[a, b]$. If f is continuous at x_0 in (a, b), then F is differentiable at x_0 and

$$F'(x) = f(x)$$

§7.1.2 Intermediate Value Theorem for Integrals.

Theorem 7.1.5: Intermediate Value Theorem for Integrals.

If f is a continuous function on $[a, b]$, then for at least one x in (a, b) we have:

$$f(x) = \int_a^b f$$

Problem 30. *Let $f(x) = x$ for rational x and $f(x) = 0$ for irrational x.*

1. *Calculate the upper and lower Darboux integrals for f on the interval $[0, b]$.*
2. *Is f integrable on $[0, b]$?*

Chapter 8

Differentiation and Integration of Power Series

Theorem 8.0.1: Weierstrass M-test

Let (M_k) be a sequence of non-negative real numbers where $\sum M_k < \infty$. If $|g_k(x)| \leq M_k$ for all x in a set S, then $\sum g_k$ converges uniformly on S

Proof. Let $\epsilon > 0$, $\exists N$ such that $n \geq m > N \Rightarrow \sum_{k=m}^{\infty} M_k < \epsilon$. Then if $n \geq m > N$ and $x \in S$,

$$\left| \sum_{k=m}^{n} g_k(x) \right| \leq \sum_{k=m}^{n} |g_k(x)| \leq \sum_{k=m}^{n} M_k < \epsilon$$

$\therefore$ by cauchy criterion on uniform cont. on S, $\sum g_k$ converges uniformly on S. ■

Problem 31. *Show that if the series $\sum g_n$ converges uniformly on a set S, then*

$$\lim_{n\to\infty} \sup\{|g_n(x)| : x \in S\} = 0$$

Proof. For $\epsilon > 0$, $\exists N$ such that $n \geq m > N \Rightarrow \left|\sum_m^n g_k(x)\right| < \epsilon$.
In particular $n > N$

$$\Rightarrow |g_n| < \epsilon \ \forall \ x \in S$$

$$\therefore \sup\{|g_n(x)| : x \in S\} \leq \epsilon$$

$$\lim_{n\to\infty} \sup\{|g_n(x)| : x \in s\} = 0$$

■

Theorem 8.0.2

Let $\sum a_n x^n$ be a power series with radius of convergence $R > 0$. If $0 < R_1 < R$, then the power series converges uniformly on $[-R_1, R_1]$ to a continuous function.

Lemma 8.0.1

f the power series $\sum a_n x^n$ has radius of convergence R, then the power series

$$\sum n a_n x^n \qquad \& \qquad \sum \frac{a_n}{n+1} x^{n+1}$$

have the same radius of convergence R.

Theorem 8.0.3

Suppose $f(x) = \sum a_n x^n$ has radius of convergence $R > 0$ Then

$$\int_0^x f(t)dt = \sum \frac{a_n}{n+1} x^{n+1} \quad \forall x < R$$

Theorem 8.0.4

Let $f(x) = \sum a_n x^n$ have radius of convergence $R > 0$. Then f is differentiable on $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} a a_n x^{n-1} \quad \forall |x| < R$$

Theorem 8.0.5: Abel's Theorem

Let $f(x) = \sum a_n x^n$ be a power series with $0 < R < \infty$. If the series converges at $x = R$, then f is continuous at $x = R$. If the series converges at $x = -R$, then f is continuous at $x = -R$.

Appendix A

Power series of some common functions[4]

Exponential Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

It converges for all $x \in \mathbb{R}$.

Natural logarithm

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots ,$$
$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots .$$

They converge for $|x| < 1$

Geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$
$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$
$$\frac{1}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{(n-1)n}{2} x^{n-2}.$$

All are convergent for $|x| < 1$. These are special cases of Binomial Series.

Binomial series The binomial series is a power series

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

whose coefficients are the generalized binomial coefficients

$$\binom{\alpha}{n} = \prod_{k=1}^{n} \frac{\alpha - k + 1}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

Trignometric Functions The usual trigonometric functions and their inverses have the following power series:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad \text{for all } x$$

$$\tan x = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n\,(1-4^n)}{(2n)!} x^{2n-1} \qquad = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \qquad \text{for } |x| < \frac{\pi}{2}$$

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} \qquad = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots \qquad \text{for } |x| < \frac{\pi}{2}$$

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} \qquad = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots \qquad \text{for } |x| \le 1$$

$$\begin{aligned}\arccos x &= \frac{\pi}{2} - \arcsin x \\ &= \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} \qquad = \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} - \cdots \qquad \text{for } |x| \le 1\end{aligned}$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \qquad = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \qquad \text{for } |x| \le 1,\ x \neq \pm i$$

All angles are expressed in radians.The numbers B_k appearing in the expansions of $\tan x$ are the Bernoulli numbers. The E_k in the expansion of $\sec x$ are Euler numbers.

Remember This text only contains important theorems and definitions from the textbook **Elementary Analysis**. The problems and solutions were discussed during the lectures by Prof. Chandrakant Aribam. And some of the problems for the book are also discussed (non-trivial problems).

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