MTH102: Analysis in One variable
Home Work No. 06
Sent on 28 February 2018

- Please do as many problems as possible.
- Please maintain a separate notebook for home work problems.
- Tutors will discuss some of these problems during tutorial sessions.
- $\mathbb{N}$ denote the set of natural numbers.
- $\mathbb{Z}$ denote the ring of integers.
- $\mathbb{Q}$ denote the field of rational numbers.
- $\mathbb{R}$ denote the field of real numbers.

(1) Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ given by $f_n(x) = (x - \frac{1}{n})^2$.
  (a) Does the sequence (f_n) converge point wise on $[0,1]$? If so, then determine the limit function.
  (b) Does the sequence (f_n) converge uniformly on $[0,1]$? If so, then prove your assertion.

(2) Repeat the above exercise for the sequence of functions $f_n : [0,1] \to \mathbb{R}$ given by $f_n(x) = x - x^n$.

(3) Repeat the above exercise for the sequence of functions $f_n : [0,\infty) \to \mathbb{R}$ given by $f_n(x) = \frac{1}{1+x^n}$.

(4) Repeat the above exercise for the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = \frac{5+3\sin^2(nx)}{\sqrt{n}}$.

(5) Let S be a subset of $\mathbb{R}$. Suppose (f_n) and (g_n) are two sequences of functions such that (f_n) converge to f and (g_n) converge to g uniformly on S. Then prove that the sequence $(f_n + g_n)$ converge to $f + g$ uniformly on S.

(6) Consider two sequences of functions $f_n,\ g_n : \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = x$ and $g_n(x) = \frac{1}{n}$. Let $f,\ g : \mathbb{R} \to \mathbb{R}$ be the functions given by $f(x) = x$ and $g(x) = 0$.
  (a) Prove that (f_n) converge to f uniformly on $\mathbb{R}$ and (g_n) converge to g uniformly on $\mathbb{R}$.
  (b) Prove that the sequence $(f_n g_n)$ does not converge to fg uniformly on $\mathbb{R}$.

(7) Let S be a subset of $\mathbb{R}$. Prove that if (f_n) is a sequence of uniformly continuous functions on S converging uniformly to f on S, then f is also uniformly continuous on S.

(8) Let (f_n) be a sequence of continuous functions on $[a,b]$ converging uniformly to a function f on $[a,b]$. Let (x_n) be a sequence in $[a,b]$ converging to a real number x. Then prove that $\lim_{n\to\infty} f_n(x_n) = f(x)$.