MTH102: Analysis in One variable
Home Work No. 02
20 January 2018

- Please do as many problems as possible.
- Please maintain a separate notebook for home work problems.
- Tutors will discuss some of these problems during tutorial sessions.
- $\mathbb{N}$ denote the set of natural numbers.
- $\mathbb{Z}$ denote the ring of integers.
- $\mathbb{Q}$ denote the field of rational numbers.
- $\mathbb{R}$ denote the field of real numbers.

(1) Determine whether the following sequences are convergent. In either case, prove your claims.
  (a) $s_n = \frac{3n+1}{5n-2}$.
  (b) $s_n = \frac{(-1)^n}{n}$.
  (c) $s_n = \frac{12n^5+73n^4-18n^2+9}{25n^5+2n^3}$.
  (d) $s_n = \frac{n}{n^2+1}$.
  (e) $s_n = (\sqrt{n^2+n} - n)$.
  (f) $s_n = \frac{1}{n}\sin(n)$.
  (g) $s_n = (-1)^n n$.
  (h) $s_n = \cos(\frac{n\pi}{3})$.

(2) Prove that every real number is the limit of a sequence of rational numbers.

(3) Prove that every real number is the limit of a sequence of irrational numbers.

(4) Let (s_n) be a sequence with $\lim s_n = s$. Let $a \in \mathbb{R}$ such that $s_n \geq a$ for all but finitely many n. Prove that $s \geq a$.

(5) Give an example of a sequence (s_n) such that $\lim |s_n|$ exists but $\lim s_n$ does not exist. Justify your answer.

(6) Let $s_1 = 1$ and $s_{n+1} = \sqrt{s_n + 1}$ for all $n \in \mathbb{N}$. Assuming that the sequence (s_n) converges, prove that $\lim s_n = \frac{1+\sqrt{5}}{2}$.

(7) Let (s_n) and (t_n) be two sequences of real numbers. Suppose that there exists $K \in \mathbb{N}$ such that $s_n \leq t_n$ for all $n > K$.
  (a) Prove that if $\lim s_n = +\infty$, then $\lim t_n = +\infty$.
  (b) Prove that if $\lim t_n = -\infty$, then $\lim s_n = -\infty$.

(8) Let (s_n) be a sequence of positive real numbers. Prove that $\lim s_n = +\infty$ if and only if $\lim \frac{1}{s_n} = 0$.

(9) Prove that

$$\lim a^n = \begin{cases} 0 & \text{if } 0 \leq |a| < 1 \\ 1 & \text{if } a = 1 \\ +\infty & \text{if } a > 1 \\ \text{Does not exist} & \text{if } a \leq -1. \end{cases}$$

(10) Prove that $\lim \frac{2^n}{n^2} = +\infty$.