MTH102: Analysis in One variable
Home Work No. 01
13 January 2018

- Please do as many problems as possible.
- Please maintain a separate notebook for home work problems.
- Tutors will discuss some of these problems during tutorial sessions.
- $\mathbb{N}$ denote the set of natural numbers.
- $\mathbb{Z}$ denote the ring of integers.
- $\mathbb{Q}$ denote the field of rational numbers.
- $\mathbb{R}$ denote the field of real numbers.

(1) Use the principle of mathematical induction to prove the following:
  (a) $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.
  (b) $n^2 > n + 1$ for all $n \in \mathbb{N}$ such that $n \geq 2$.
  (c) $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$ for all $n \in \mathbb{N}$.

(2) Prove that $(2 + 5^{1/3})^{1/2}$, $(2 + 2^{1/2})^{1/2}$ and $(5 - 3^{1/2})^{1/3}$ are not rational numbers.

(3) Prove that $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

(4) Prove that $|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$ for any n real numbers.

(5) Prove that $|a - b| \leq c$ if and only if $b - c \leq a \leq b + c$.

(6) Let $a, b \in \mathbb{R}$. Prove that if $a \leq c$ for all $c > b$, then $a \leq b$.

(7) Prove that the set of irrational numbers is dense in the set of real numbers.

(8) Determine whether the following sets are bounded or not. If so, then determine their supremums and infimums. Do these numbers lie in the given sets?
  (a) $A = \{r \in \mathbb{Q} \mid r^2 < 4\}$.
  (b) $B = \{1 - \frac{1}{3^n} \mid n \in \mathbb{N}\}$.
  (c) $C = \{n^{(-1)^n} \mid n \in \mathbb{N}\}$.
  (d) $D = \{\frac{1}{n} \mid n \in \mathbb{N}\}$.

(9) Let A be a subset of $\mathbb{R}$ and $b \in \mathbb{R}$ a fixed real number. Suppose that $a < b + \epsilon$ for all $a \in A$ and each $\epsilon > 0$. Then prove that b is an upper bound for A.

(10) Suppose that A, B are non-empty sets of real numbers such that $x \leq y$ for all $x \in A$ and $y \in B$. Then prove that $\sup A \leq \inf B$.

(11) Write down a proof of the Binomial Theorem using the principle of mathematical induction.