

**IISER Mohali**  
**MTH102: Analysis in One Variable**  
**Homework No. 09**  
**To be discussed during tutorial on March 29, 2016**

- Please solve all the problems.
- Tutors will discuss **tutorial problems** during tutorial sessions.
- If time permits, tutors may also discuss **extra problems** during tutorial sessions.

**Tutorial Problems:**

- (1) Let  $f(x) = |x-1| + |x-2|$  for all  $x \in \mathbb{R}$ . Determine the set of points at which  $f$  is not differentiable.  
(2) Let

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

- (a) Show that  $f$  is differentiable at all  $x \in \mathbb{R}$ . Determine the function  $f' : \mathbb{R} \rightarrow \mathbb{R}$ .  
(b) Is the function  $f'$  continuous on  $\mathbb{R}$ ?  
(c) Is the function  $f'$  differentiable on  $\mathbb{R}$ ?  
(3) Let

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

- (a) Prove that  $f$  is continuous at  $x = 0$ .  
(b) Prove that  $f$  is differentiable at  $x = 0$  and find the derivative.  
(c) What happens at points  $x \neq 0$ ?  
(4) Let  $f$  be a differentiable function on  $\mathbb{R}$ . Further, suppose that  $f(0) = 0$ ,  $f(1) = 1$  and  $f(2) = 1$ .  
(a) Show that  $f'(x) = \frac{1}{2}$  for some  $x \in (0, 2)$ .  
(b) Show that  $f'(x) = \frac{1}{7}$  for some  $x \in (0, 2)$ .  
Hint: Use Mean Value Theorem.

- (5) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function with  $|f(x) - f(y)| \leq (x - y)^2$  for all  $x, y \in \mathbb{R}$ . Then prove that  $f$  is a constant function.

Hint: Show that  $f$  is differentiable with  $f'(x) = 0$  for all  $x \in \mathbb{R}$ .

- (6) Let  $f$  be a twice differentiable function on an open interval  $(a, b)$  such that  $f''(x) = 0$  for all  $x \in (a, b)$ . Then prove that  $f$  has the form  $f(x) = \alpha x + \beta$  for some  $\alpha, \beta \in \mathbb{R}$ .

Hint: Use a consequence of the Mean Value Theorem.

- (7) Let  $f$  and  $g$  be two differentiable functions on  $\mathbb{R}$ . Suppose that  $f'(x) \leq g'(x)$  for all  $x \in \mathbb{R}$  and  $f(0) = g(0)$ . Then prove that  $f(x) \leq g(x)$  for all  $x \geq 0$ .

Hint: Use properties of increasing or decreasing functions.

**Extra Problems:**

- (1) Let  $f(x) = |\sin(x)|$  for all  $x \in \mathbb{R}$ . Give the exact set of points at which  $f$  is not differentiable.  
Hint: Drawing a graph of  $f(x)$  will be helpful.  
(2) Let  $f(x) = \sin(|x|)$  for all  $x \in \mathbb{R}$ . Give the exact set of points at which  $f$  is not differentiable.  
Hint: You may assume that  $\sin(x)$  is a differentiable function with  $\cos(x)$  as its derivative.  
(3) Let  $f(x) = x^{1/3}$  for  $x \in \mathbb{R}$ . Use the definition of derivative to prove that  $f'(x) = \frac{1}{3}x^{-2/3}$  for all  $x \neq 0$ . Is the function  $f$  differentiable at  $x = 0$ ?  
(4) Suppose that  $f$  is differentiable at a point  $a \in \mathbb{R}$ . Then prove that

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

- (5) Let  $f$  be a thrice differentiable function on an open interval  $(a, b)$  such that  $f'''(x) = 0$  for all  $x \in (a, b)$ . What form does  $f$  have? Prove your claim.