IISER Mohali
MTH102: Analysis in One Variable
Homework Sheet No. 08
To be discussed during tutorial on March 11, 2016

- Please solve all the problems.
- Tutors will discuss **tutorial problems** during tutorial sessions.
- If time permits, tutors may also discuss **extra problems** during tutorial sessions.

Tutorial Problems:

(1) Let $f_n(x) = (x - \frac{1}{n})^2$ for $x \in [0, 1]$.
  (a) Does the sequence (f_n) converge point wise on the set $[0, 1]$? If so, then give the limit function.
  (b) Does the sequence (f_n) converge uniformly on the set $[0, 1]$? If so, then prove your assertion.

(2) Repeat the above exercise for $f_n(x) = x - x^n$ for $x \in [0, 1]$.

(3) Repeat the above exercise for $f_n(x) = \frac{1}{1+x^n}$ for $x \in [0, \infty)$.

(4) Let $S \subseteq \mathbb{R}$. Prove that if a sequence (f_n) converge to f uniformly on S and a sequence (g_n) converge to g uniformly on S, then the sequence $(f_n + g_n)$ converge to $f + g$ uniformly on S.
Hint: Use the definition of uniform convergence and $\frac{\epsilon}{2}$ trick.

(5) Let $f_n(x) = x$ and $g_n(x) = \frac{1}{n}$ for all $x \in \mathbb{R}$. Let $f(x) = x$ and $g(x) = 0$ for all $x \in \mathbb{R}$.
  (a) Prove that (f_n) converge to f uniformly on $\mathbb{R}$ and (g_n) converge to g uniformly on $\mathbb{R}$.
  (b) Prove that the sequence $(f_n g_n)$ does not converge to fg uniformly on $\mathbb{R}$.

(6) Let $S \subseteq \mathbb{R}$. Prove that if (f_n) is a sequence of uniformly continuous functions on S converging uniformly to f on S, then f is also uniformly continuous on S.
Hint: Use the definition of uniform convergence, uniform continuity and $\frac{\epsilon}{3}$ trick.

Extra Problems:

(1) Repeat the tutorial problem (1) for $f_n(x) = \frac{5+3\sin^2(nx)}{\sqrt{n}}$ for $x \in \mathbb{R}$.

(2) Let (f_n) be a sequence of continuous functions on $[a, b]$ converging uniformly to a function f on $[a, b]$. Let (x_n) be a sequence in $[a, b]$ converging to real number x. Prove that $\lim_{n\to\infty} f_n(x_n) = f(x)$.
Hint: Use the definition of uniform convergence, continuous function and $\frac{\epsilon}{3}$ trick.