

**IISER Mohali**  
**MTH102: Analysis in One Variable**  
**Homework No. 02**  
**To be discussed during tutorial on January 22, 2016**

- Please solve all the problems.
- **Tutorial Problems** will be discussed during tutorial sessions.
- If time permits, tutors may also discuss **Extra Problems** during tutorial sessions.

**Tutorial Problems:**

- (1) Let  $S$  and  $T$  be non-empty bounded subsets of  $\mathbb{R}$ . Let  $S + T = \{s + t \mid s \in S \text{ and } t \in T\}$ .
  - (a) Prove that if  $S \subseteq T$ , then  $\inf(T) \leq \inf(S) \leq \sup(S) \leq \sup(T)$ .
  - (b) Prove that  $\sup(S \cup T) = \max\{\sup(S), \sup(T)\}$ .
  - (c) Prove that  $\inf(S \cup T) = \min\{\inf(S), \inf(T)\}$ .
  - (d) Prove that  $\sup(S + T) = \sup(S) + \sup(T)$ .
  - (e) Prove that  $\inf(S + T) = \inf(S) + \inf(T)$ .
- (2) Let  $\mathbb{I}$  be the set of all irrational numbers. Prove that if  $a < b$  are two real numbers, then there exists  $x \in \mathbb{I}$  such that  $a < x < b$ .
- (3) Prove that if  $0 < a$  is a real number, then there exists  $n \in \mathbb{N}$  such that  $\frac{1}{n} < a < n$ .
- (4) Determine the limits of the following sequences and prove your claims:
  - (a)  $\lim_{n \rightarrow \infty} \frac{3n+1}{5n-2}$ .
  - (b)  $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n}$ .
- (5) Give an example of a sequence of rational numbers converging to an irrational number.
- (6) Let  $(s_n)$  be a sequence of non-negative real numbers and let  $\lim s_n = s$ . Prove that  $\lim \sqrt{s_n} = \sqrt{s}$ .
- (7) Let  $(s_n)$  be a sequence such that  $\lim s_n = s$ . Let  $a \in \mathbb{R}$  and  $s_n \geq a$  for all but finitely many  $n$ . Prove that  $s \geq a$ .

**Extra Problems:**

- (1) Let  $S$  and  $T$  be non-empty subsets of  $\mathbb{R}$ , not necessarily bounded. Prove that if  $S \subseteq T$ , then  $\inf(T) \leq \inf(S) \leq \sup(S) \leq \sup(T)$ .
- (2) Determine the limits of the following sequences and prove your claims:
  - (a)  $\lim_{n \rightarrow \infty} \frac{n}{n^2+1}$ .
  - (b)  $\lim_{n \rightarrow \infty} (\sqrt{n^2 + n} - n)$ .
  - (c)  $\lim_{n \rightarrow \infty} \frac{1}{n} \sin(n)$ .
- (3) Prove that the following sequences do not converge.
  - (a)  $(-1)^n n$ .
  - (b)  $\sin\left(\frac{n\pi}{3}\right)$ .
  - (c)  $\cos\left(\frac{n\pi}{3}\right)$ .
- (4) Give an example of a sequence of irrational numbers converging to a rational number.
- (5) Give an example of a sequence  $(s_n)$  such that  $\lim |s_n|$  exists but  $\lim s_n$  does not exist.