MTH102: Analysis in One variable
Home Work Problems: 01
10 January 2016

- Please do all the problems.
- Maintain a separate notebook for home work problems.
- **Tutorial Problems** will be discussed during the tutorials.
- If time permits, the tutors may discuss **Extra Problems** during the tutorials.

Tutorial Problems:

(1) Using the principle of mathematical induction, prove that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.

(2) Prove that $(2 + 5^{1/3})^{1/2}$ is not a rational number.

(3) Prove that if $0 < a < b$, then $0 < b^{-1} < a^{-1}$ for all $a, b \in \mathbb{R}$.

(4) Prove that $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

(5) Prove that $|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$ for any set of n real numbers for each $n \in \mathbb{N}$.

(6) Prove that $|a - b| \leq c$ if and only if $b - c \leq a \leq b + c$.

(7) Let $a, b \in \mathbb{R}$. Prove that if $a \leq c$ for all $c > b$, then $a \leq b$.

(8) Determine whether the following sets are bounded above and bounded below. If so, then give an upper and a lower bound.
   (a) $\{r \in \mathbb{Q} \mid r^2 < 4\}$.
   (b) $\{1 - \frac{1}{3^n} \mid n \in \mathbb{N}\}$.
   (c) $\{n^{(-1)^n} \mid n \in \mathbb{N}\}$.

Extra Problems:

(1) Using the principle of mathematical induction, prove that $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$ for all $n \in \mathbb{N}$.

(2) Using the principle of mathematical induction, prove that $n^2 > n + 1$ for all $n \in \mathbb{N}$ such that $n \geq 2$.

(3) Prove that $(2 + 2^{1/2})^{1/2}$ and $(5 - 3^{1/2})^{1/3}$ are not a rational numbers.

(4) Let A be a subset of $\mathbb{R}$ and let $b \in \mathbb{R}$ a fixed real number. Suppose that for all $a \in A$ and $\epsilon > 0$, we have $a < b + \epsilon$. Then prove that b is an upper bound for A.

(5) Using the principle of mathematical induction, write down a proof of the binomial theorem.