IISER Mohali
MTH102: Analysis in One Variable
Homework No. 06
To be discussed during tutorial on February 26, 2016

- Please solve all the problems.
- Tutors will discuss **tutorial problems** during tutorial sessions.
- If time permits, tutors may also discuss **extra problems** during tutorial sessions.

Tutorial Problems:

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ \frac{1}{10^{10}} & \text{if } x < 0 \end{cases}$$

Show that f is not continuous at 0.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^3 \cos(\frac{1}{x^2}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Use the $\epsilon - \delta$ definition to show that f is continuous at 0. Is f continuous at $x \neq 0$ and why?

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number} \\ 0 & \text{if } x \text{ is an irrational number} \end{cases}$$

Show that f is not continuous at any $x \in \mathbb{R}$.

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is a rational number} \\ 0 & \text{if } x \text{ is an irrational number} \end{cases}$$

Show that f is continuous at 0 and discontinuous at every other point of $\mathbb{R}$.

(5) Give an example of an unbounded continuous function on the open interval (a, b).

(6) Let $f, g : [a, b] \to \mathbb{R}$ be two continuous functions such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Then show that there exists a point $x_0 \in [a, b]$ such that $f(x_0) = g(x_0)$.

(7) Use the $\epsilon - \delta$ definition of uniform continuity to show that the function $f : [0, 2] \to \mathbb{R}$ defined by $f(x) = x^2$ is uniformly continuous.

(8) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not uniformly continuous.

Extra Problems:

(1) Prove that $x = \cos(x)$ for some $x \in (0, \frac{\pi}{2})$.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that f is not continuous at 0.

(3) Use the $\epsilon - \delta$ definition of uniform continuity to show that the function $f : [0, 2] \to \mathbb{R}$ defined by $f(x) = \frac{x}{x+2}$ is uniformly continuous on $[0, 2]$.

(4) Let $f : S \to \mathbb{R}$ be a uniformly continuous function on a subset S of $\mathbb{R}$. Show that if (x_n) is a Cauchy sequence in S, then $\big(f(x_n)\big)$ is also a Cauchy sequence in $\mathbb{R}$.