

**IISER Mohali**  
**MTH102: Analysis in One Variable**  
**Homework No. 05**  
**To be discussed during tutorial on February 19, 2016**

- Please solve all the problems.
- Tutors will discuss **tutorial problems** during tutorial sessions.
- If time permits, tutors may also discuss **extra problems** during tutorial sessions.

**Tutorial Problems:**

- (1) Determine whether the following series converge.
  - (a)  $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$
  - (b)  $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$
  - (c)  $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$
- (2) Suppose that  $\sum_{n=1}^{\infty} a_n = A$  and  $\sum_{n=1}^{\infty} b_n = B$ , where  $A$  and  $B$  are real numbers. Then prove the following:
  - (a)  $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$ .
  - (b)  $\sum_{n=1}^{\infty} k a_n = kA$  for all  $k \in \mathbb{R}$ .
  - (c) Can we say that  $\sum_{n=1}^{\infty} a_n b_n = AB$ ?
- (3) Prove that if  $\sum_{n=1}^{\infty} |a_n|$  converges and  $(b_n)$  is a bounded sequence, then  $\sum_{n=1}^{\infty} a_n b_n$  converges. Hint: Use the Cauchy criteria.
- (4) Let  $f : (a, b) \rightarrow \mathbb{R}$  be a continuous function such that  $f(r) = 0$  for each rational number  $r \in (a, b)$ . Then prove that  $f(x) = 0$  for each  $x \in (a, b)$ .
- (5) Let  $f(x) = \frac{1}{x} \sin(\frac{1}{x^2})$  for  $x \neq 0$  and  $f(0) = 0$ . Show that  $f$  is not continuous at 0.

**Extra Problems:**

- (1) Let  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  be two series such that  $a_n = b_n$  for all but finitely many  $n \in \mathbb{N}$ . Then prove the following:
  - (a)  $\sum_{n=1}^{\infty} a_n$  converges if and only if  $\sum_{n=1}^{\infty} b_n$  converges.
  - (b)  $\sum_{n=1}^{\infty} a_n$  diverges if and only if  $\sum_{n=1}^{\infty} b_n$  diverges.
- (2) Prove that  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = |x| + 2x$  is a continuous function.
- (3) Give two continuous functions  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f \circ g \neq g \circ f$ .