Assignment 7

1. (i) $Z(G) = \{ z \in G : zg = gz \ \forall g \in G \}$

for $z \in Z(G)$ and $g \in G$,

$$zg = gz \Rightarrow gzg^{-1} = z \in Z(G) \ \forall g \in G, z \in Z(G)$$

$\Rightarrow Z(G)$ is a normal subgp of G.

elements of the quotient group $G/Z(G)$ are of the form $Z(G)g$, $g \in G$.

(ii) $[GG] = \langle aba^{-1}b^{-1} : a, b \in G \rangle$
= subgroup of G generated by elts of the form $aba^{-1}b^{-1}$ for $a, b \in G$.

To show: $[G,G]$ is a normal subgp of G, it is sufficient to prove that for $a, b \in G$, and any $g \in G$,

$$g\,aba^{-1}b^{-1}g^{-1} \in [G,G]$$

Note that

$$gab\,a^{-1}b^{-1}g^{-1} = ga(g^{-1}g)\,b\,(g^{-1}g)\,a^{-1}(g^{-1}g)\,b^{-1}(g^{-1}g)g^{-1}$$
$$= (gag^{-1})(gbg^{-1})(ga^{-1}g^{-1})(gb^{-1}g^{-1}).$$

By closure in G, for $a, g, b \in G$,

$A = gag^{-1}$, $B = gbg^{-1} \in G$, and $ga^{-1}g^{-1} = (gag^{-1})^{-1}$

$gb^{-1}g^{-1} = (gbg^{-1})^{-1}$

$\therefore\ gaba^{-1}a^{-1}b^{-1}g^{-1}$

$= ABA^{-1}B^{-1} \in [G,G]$.

$\therefore [G,G]$ is normal in G.

Claim: $\dfrac{G}{[G,G]}$ is abelian.

~~Let~~ Any elt of $G/[G,G]$ is $a[G,G]$.

Consider $a[G,G] \cdot b[G,G]$

$= ab[G,G]$ ($\because [G,G]$ is $\trianglelefteq G$)

$\cancel{a^{-1}b^{-1}}$ $= (ba)(ba)^{-1}(ab)[G,G]$

$= (ba)(a^{-1}b^{-1})(ab)[G,G]$

$= ba\,(a^{-1}b^{-1}ab)[G,G]$ — ⊛

Note $a^{-1}b^{-1}ab \in [G,G]$

$\therefore (a^{-1}b^{-1}ab)[G,G] = [G,G]$.

From ⊛ we get

$\therefore\ a[G,G] \cdot b[G,G] = (ba)(a^{-1}b^{-1}ab)[G,G]$

$= (ba)[G,G]$

$= b[G,G] \cdot a[G,G]$

($\because [G,G]$ is $\trianglelefteq G$).

this proves that the quotient group $G/[G,G]$ is abelian.

2. (i) Suppose K is normal in G and K is a subgroup of a normal subgroup N of G.

Then K is normal in N.

This is clear, since $K \trianglelefteq G$ $\therefore$ $gkg^{-1} \in K$ for all $g \in G$, $k \in K$.

In particular for $n \in N$ and $k \in K$, $nkn^{-1} \in K$, which implies that K is normal in N.

However if K is normal in N, and N is normal in G then K need not be normal in G.

For examples

$N = \{(12)(34), (13)(24), (14)(32), e\}$

is a normal subgp of $G = S_n$, and

$K = \{e, (12)(34)\}$ is a normal subgp of N.

But K is not a normal subgroup of S_n.

Check that

$$\left.\begin{array}{l} (13)K = \{(13), (1234)\} \\ K(13) = \{(13), (1432)\} \end{array}\right\} \Rightarrow K(13) \neq (13)K$$

Hence K is not a normal subgp of $G = S_n$.

2(ii). Let $\phi : G \longrightarrow G/N$ be the natural map

$$\phi(g) = Ng.$$

Claim: ϕ is onto group homomorphism.

Any elt of G/N is of the form Ng for $g \in G$.

$\therefore$ by defination ϕ is onto.

Now $\phi(g_1 g_2) = N g_1 g_2$

$$= Ng_1 Ng_2 \quad (\because N \trianglelefteq G)$$

$$= \phi(g_1)\phi(g_2).$$

This shows that ϕ is ~~an~~ a gp homomorphism.

Kernel $\phi = \{ g \in G \mid \phi(g) = e_{G/N} \}$

The identity element of G/N is N.

$\therefore \phi(g) = e_{G/N} \Rightarrow \phi(g) = N$

$\Rightarrow Ng = N \Rightarrow g \in N.$

$\therefore \operatorname{Ker}\phi \subseteq N.$

On the other hand if $n \in N$, $\phi(n) = Nn = N.$

$\Rightarrow N \subseteq \operatorname{Ker}\phi.$

$\Rightarrow \operatorname{Ker}\phi = N.$

2(iii). Let $K \trianglelefteq G$, $N \trianglelefteq G$ and let

$\phi : G/K \longrightarrow G/N$ be the natural

grp homomorphism given by

$\phi(Kg) = Ng, \quad \forall g \in G.$

For $k_1 \in K$, $Kk_1 = K$ = identity element of G/K, ⑤

~~But~~ and under the gp homomorphism

$\phi(Kk_1) = \phi(K) = e_{G/N} = N$. — ①

also $\phi(Kk_1) = Nk_1$ — ②.

① and ② together imply that

$$Nk_1 = N \quad \forall k_1 \in K$$

$$\Rightarrow k_1 \in N \quad \text{i.e. } K \subseteq N.$$

2 iv. Let K be a subgp of G. $N \trianglelefteq G$ (normal subgp of G)

Claim: $NK = \{nk : n \in N, k \in K\}$ is a subgp of G.

To prove NK is a subgp of G, it suffices to show that if $n_1k_1, n_2k_2 \in NK$, then

$$n_1k_1(n_2k_2)^{-1} \in NK.$$

But $n_1k_1(n_2k_2)^{-1} = n_1k_1k_2^{-1}n_2^{-1}$.

$\because$ N is normal in G, $\therefore$ for $k_1k_2^{-1} \in G$,

$(k_1k_2^{-1})N = N(k_1k_2^{-1})$

$\Rightarrow \exists\, n' \in N$ st $k_1k_2^{-1}n_2^{-1} (\in k_1k_2^{-1}N)$

$= n'k_1k_2^{-1} (\in Nk_1k_2^{-1})$

$\Rightarrow (n_1k_1)(n_2k_2)^{-1} = n_1n'k_1k_2^{-1} \in NK$ ($\because n_1, n' \in N$, $k_1k_2^{-1} \in K$)

3. G - group, $x \in G$ fixed element.

Let $\phi_x : G \longrightarrow G$ be defined by

$$g \longmapsto x g x^{-1}.$$

<u>Claim:</u> ϕ_x is a group homomorphism.

Let $g_1, g_2 \in G$, then

$$\begin{aligned}
\phi_x(g_1 g_2) &= x g_1 g_2 x^{-1} \\
&= (x g_1)\cdot(e)(g_2 x^{-1}) \\
&= (x g_1)(x^{-1}x)(g_2 x^{-1}) \\
&= (x g_1 x^{-1})(x g_2 x^{-1}) \\
&= \phi_x(g_1)\,\phi_x(g_2).
\end{aligned}$$

Hence ϕ_x is a group homomorphism.

Let $g \in \operatorname{Ker} \phi_x$

$\Rightarrow \quad \phi_x(g) = e \quad \Rightarrow \quad x g x^{-1} = e$ —(1)

Premultiply both sides of (1) by x^{-1} and postmultiply by x we get

$$(x^{-1})(x g x^{-1})(x) = x^{-1} . e x$$

$$\Rightarrow (x^{-1}x) g (x^{-1}x) = x^{-1}x = e$$

$$\Rightarrow \quad g = e \qquad \therefore \quad \operatorname{Ker} \phi_x = \{e\}.$$

This shows that the group homomorphism ϕ_x is injective.

Let $y \in G$ be an arbitrary element.

Then observe that

$$\phi_x(x^{-1}yx) = x(x^{-1}yx)x^{-1}$$
$$= (xx^{-1})y(xx^{-1}) = y.$$

$\therefore$ for every $y \in G$, $x^{-1}yx \in G$ is st $\phi_x(x^{-1}yx) = y$ which shows that ϕ_x is onto.

Hence $\phi_x : G \to G$ is a one-one, onto group homomorphism from $G \to G$. i.e ϕ_x is an isomorphism. (An isomorphism $\phi : G \to G$ from G to itself is called an <u>automorphism</u> of G.)

4(i). G - group, let $L_x : G \to G$ be defined by $L_x(g) = xg$.

Let $g_1, g_2 \in G$, then

$$L_x(g_1 g_2) = x g_1 g_2 = (x g_1) g_2$$
$$\neq L_x(g_1) L_x(g_2)$$
$$= xg_1(xg_2).$$

Hence L_x is not a group homomorphism.

Let $y \in G$ be an arbitrary element.

Then $L_x(x^{-1}y) = x(x^{-1}y) = (xx^{-1})y$
$= y.$

Hence L_x is an onto map. — (1)

Let $L_x(g_1) = L_x(g_2)$

$\Rightarrow xg_1 = xg_2$ — (2)

Premultiplying both sides of (2) by x^{-1} we get

$$x^{-1}(xg_1) = x^{-1}(xg_2)$$

$$\Rightarrow (x^{-1}x)g_1 = (x^{-1}x)g_2$$

$$\Rightarrow e.g_1 = e.g_2 \Rightarrow g_1 = g_2$$

This shows that L_x is a one-one map. — (3)

From (1) and (3) we conclude that L_x is a bijection.

(ii) Let $\mathbb{L}_G = \{L_x : x \in G\}$

Claim: $\mathbb{L}_G$ is a group wrt composition of ~~G~~ maps.

let $L_{x_1}, L_{x_2} \in \mathbb{L}_G$.

Then for any $y \in G$,

$$L_{x_1} \cdot L_{x_2}(y) = L_{x_1}(x_2 y) = (x_1 x_2) y$$

$\because$ for $x_1, x_2 \in G$, $x_1 x_2 \in G$ $\quad = L_{x_1 x_2}(y)$.

$\therefore L_{x_1 x_2} \in \mathbb{L}_G$. This shows that $\mathbb{L}_G$ is closed under composition of maps.

· Using associativity in G it is easy to check that for $x_1, x_2, x_3 \in G$,

$$\begin{aligned} L_{x_1} \cdot (L_{x_2} \cdot L_{x_3})(y) &= L_{x_1}(x_2 x_3 y) \\ &= x_1 (x_2 x_3) y \\ &= (x_1 x_2) x_3 y \\ &= L_{x_1 x_2}(L_{x_3} y) \\ &= (L_{x_1} \cdot L_{x_2}) L_{x_3}(y) \end{aligned}$$

$\therefore (\mathbb{L}_G, \cdot)$ is associative.

For $x \in G$, consider $L_{x^{-1}} \in \mathbb{L}_G$.

Clearly $L_x \cdot L_{x^{-1}}(y) = x x^{-1}(y)$
$= e \cdot y = y = (x^{-1}x) y$
$= L_{x^{-1}} L_x(y)$
$\forall y \in G$

$$\Rightarrow L_x \cdot L_{x^{-1}} = L_{x^{-1}} \cdot L_x = I_{\mathbb{L}_G} = L_e$$

$$L_e . L_x(y) = L_e(xy) = e.xy$$
$$= x.ey = xy = L_x(y)$$
$$= L_x . L_e(y).$$

$$\therefore L_e . L_x = L_x = L_x . L_e \quad \forall x \in G.$$

This shows that $\mathbb{L}_G$ is a gp wrt composition of maps.

Let $L : G \longrightarrow \mathbb{L}_g$ we defined by
$$g \longmapsto L_g.$$

~~Clea~~ Then for $g_1, g_2 \in G$,

$$L(g_1 g_2) = L_{g_1 g_2} = L_{g_1} L_{g_2} \text{ (by previous calculation)}$$
$$= L(g_1) L(g_2)$$

$\therefore$ L is a group homomorphism.

• $\because$ every elt of $\mathbb{L}_G$ is of the form $L_g, g \in G$, L is clearly onto.

• Suppose $L(g_1) = L(g_2)$ for some $g_1, g_2 \in G$.

then $L_{g_1} = L_{g_2}$

$$\Rightarrow L_{g_1}(x) = L_{g_2}(x) \quad \forall x \in G$$

In particular for $x = e$,

$$L_{g_1}(e) = L_{g_2}(e)$$
$$\Rightarrow g_1 = g_2 \quad \therefore L \text{ is one-one.}$$

Hence $L : G \longrightarrow \mathbb{L}_G$ is a one-one, onto group ~~isompor~~ isomorphism, which implies that L is an ~~gp~~ isomorphism of groups.

5. Let $H \leq G$ (subgp).

Then $Hx = Hy$ for $x, y \in G$ iff $xy^{-1} \in H$.

Suppose $Hx = Hy$

$\Rightarrow$ The sets $Hx = \{hx : h \in H\}$ is equal to the set $Hy = \{hy : h \in H\}$.

Hence given $hx \in Hx$, $\exists$ some $h' \in H$

st $hx = h'y$ — ①.

Postmultiplying both sides of ① by y^{-1} and Premultiplying both sides of ① by h^{-1} we get

$$h^{-1}(hx)y^{-1} = h^{-1}(h'y)y^{-1}$$

$$\Rightarrow (h^{-1}h)xy^{-1} = (h^{-1}h')(yy^{-1})$$

$$\Rightarrow xy^{-1} = h^{-1}h'$$

$\because h, h' \in H$, RHS is in H.

i.e $xy^{-1} \in H$.

Conversely, assume that $xy^{-1} \in H$.

This implies that $\exists$ $h \in H$ st

$$xy^{-1} = h. \quad \text{--- (2)}.$$

Postmultiplying by sides of (2) by y we get,

$$(xy^{-1})y = hy$$

$$\Rightarrow x = hy. \quad \text{--- (3)}.$$

$\because hy \in Hy$, by (3) we see that $x \in Hy$.

On the other hand, since $H \leq G$, $e \in H$.

and $Hx \ni e.x = x$, which implies

that $x \in Hx \cap Hy$. --- (4)

But we know that two right cosets of H in G are either equal or they are disjoint.

Hence it follows from (4) that $Hx = Hy$.

Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad \neq 0, \ a, b, d \in \mathbb{R} \right\}$

$$H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}$$

Then for $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G$, $g^{-1} = \begin{pmatrix} \frac{d}{ad} & -b/ad \\ 0 & \frac{a}{ad} \end{pmatrix}$

$$g \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g^{-1} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{d}{ad} & -\frac{b}{ad} \\ 0 & \frac{a}{ad} \end{pmatrix}$$

$$\Rightarrow g\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g^{-1} = \begin{pmatrix}a & ax+b\\ 0 & d\end{pmatrix}\begin{pmatrix}\frac{d}{ad} & -\frac{b}{ad}\\ 0 & \frac{a}{ad}\end{pmatrix}$$

$$= \begin{pmatrix}1 & -\frac{b}{d}+\frac{ax+b}{d}\\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & \frac{ax}{d}\\ 0 & 1\end{pmatrix} \in H.$$

$\because$ any arbitrary elt of H is of the form $h=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$ and any arbitrary elt of G is of the form $g=\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}$, we see that $ghg^{-1} \in H \ \forall\ h\in H,\ g\in G$.

Hence H is a normal subgp of G.

Now observe that

$$\begin{pmatrix}a & b\\ 0 & d\end{pmatrix} = \begin{pmatrix}1 & b/d\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}^{-1} = \begin{pmatrix}1 & b/d\\ 0 & 1\end{pmatrix} \in H.$$

Hence by 1st part we see that

$$H\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = H\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

This shows that any general element of the quotient group $G/H = \{Hg : g \in G\}$ is of the form $H\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$.

Let $H\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, H\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in G/H$.

Then
$$H\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} H\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = H\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

($\because$ H is normal in G)

$$= H\begin{pmatrix} ax & 0 \\ 0 & dy \end{pmatrix}$$

$$= H\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

($\because$ diagonal matrices commute)

$$= H\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} H\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

($\because$ H is normal in G)

$$\Rightarrow H\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} H\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = H\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} H\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

$\Rightarrow$ Hence G/H is ~~abelian~~ abelian.