MTH 101 - Symmetry
Assignment 7

Notes: Given a group G, a subgroup N of G is said to be **normal** if for all $n \in N$ and $g \in G$, $gng^{-1} \in N$.
If N is a normal subgroup of G, then the set of right cosets of N in G forms a group called the **quotient group.**
We denote this group by G/N.

1. Let G be a group. Then prove that the following are normal subgroups of G.
   i. $Z(G) = \{z \in G : zg = gz, \text{ for all } g \in G\}$.
   ii. $[GG] = \{aba^{-1}b^{-1} : a, b \in G\}$

2. Let N be a normal subgroup of a group G. Then prove the following.
   i. If K is a normal subgroup of N, then K is normal in G.
   ii. Let $\phi : G \to G/N$ be the map given by $g \mapsto Ng$. Show that ϕ is an onto group homomorphism.
   iii. Let K be a normal subgroup of G and $\phi : G/K \to G/N$ be a group homomorphism. Then prove that $K \subseteq N$.
   iv. Let K be a subgroup of G. Then $NK = \{nk : n \in N, k \in K\}$ is a subgroup of G.

3. Let G be a group and fix an element $x \in G$. Let $\phi_x : G \to G$ be the map defined by $g \mapsto xgx^{-1}$. Prove that ϕ_x is a group isomorphism from G to G.

4. Let G be a group and for every $x \in G$, let $L_x : G \to G$ be the map defined by $g \mapsto xg$.
   i. Prove that L_x is a bijection from G to G which is not a group homomorphism.
   ii. Let $\mathbb{L}_G = \{L_x : x \in G\}$. Prove that $\mathbb{L}_G$ is a group with respect to composition of maps and the map $L : G \to \mathbb{L}_G$ defined by $g \mapsto L_g$ is a group isomorphism.

5. Let H be a subgroup of G. Then prove that $Hx = Hy$ for $x, y \in G$ if and only if $xy^{-1} \in H$. In particular when G is the set of all real 2×2 matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ such that $ad \neq 0$ and $H = \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R}\}$ then show that H is a normal subgroup of G, $H\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = H\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ and hence G/H is abelian.