MTH 101 - Symmetry
Assignment 3

1. Let A be a $m \times n$ matrix and B be a $n \times k$ matrix. Show that the rows of $C = AB$ are linear combinations of rows of B and the columns of C are linear combinations of columns of A.

2. Let A be an $n \times n$ matrix such that the trivial solution $X = 0$ is the only solution for the system of linear equations $AX = 0$. Then prove the following.
   (a) A is row equivalent to the identity matrix.
   (b) A is invertible.
   (c) If B is a $n \times n$ matrix such that $AB = 0$. Then $B = 0$.

3. Let A be an $n \times n$ matrix. If A is not invertible then prove that there exists infinity many $n \times n$ matrices B such that $AB = 0$.

4. An $n \times n$ matrix A is called *upper-triangular* if and only if every entry below the main diagonal is 0. Prove that an upper triangular matrix is invertible if and only if every entry on its main diagonal is different from 0.

5. Given functions $f : A \to B$, $g : B \to C$, $h : C \to D$ show that the following hold.
   (a) If f and g are one-one then $g \circ f$ is one-one.
   (b) If f and g are onto then $g \circ f$ is onto.
   (c) If f is one-one and onto then there exists a map $f' : B \to A$ such that $f' \circ f = id_A$ and $f \circ f' = id_B$, where $id_A : A \to A$ denotes the identity map on A.
   (d) Suppose A is a finite set and $\alpha : A \to A$ is a one-one map on A. Then show that α is also onto.
   (e) Suppose A is a finite set and $\alpha : A \to A$ is onto. Then show that α is also one-one.

6. Determine if the following functions $f : \mathbb{Z} \to \mathbb{Z}$ are one-one, onto, both one-one and onto.
   (a) $f(n) = n + k$.
   (b) $f(n) = \begin{cases} n & \text{if } n < 0 \\ n+1 & \text{if } n \geq 0 \end{cases}$
   (c) $f(n) = \begin{cases} n+1 & \text{if } n \leq 0 \\ n & \text{if } n > 0 \end{cases}$
   (d) $f(n) = n^2$.
   (e) $f(n) = |n|$.