Assignment 11

1. $v_1 = (1,-1), \quad v_2 = (2,-1), \quad v_3 = (-3,2)$

$w_1 = (1,0), \quad w_2 = (0,1), \quad w_3 = (1,1)$

If $Tv_1 = w_1$ and $Tv_2 = w_2$

then $Tv_1 + Tv_2 = w_1 + w_2 = w_3$

$\Rightarrow T(v_1+v_2) = w_3.$

~~But~~ On the other hand

$(-3,2) = v_3 = -v_1 - v_2$
$= (-1,1) + (-2,1).$

$\therefore Tv_3 = -w_1 - w_2 = -w_3$

Hence there cannot exist a linear transformation $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ st $T(v_i) = w_i$, $i=1,2,3$.

2. Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be st

$T(1,0) = (a,b)$ and $T(0,1) = (c,d)$.

Then for any $(x_1,x_2) \in \mathbb{R}^2$

$$
\begin{aligned}
T(x_1,x_2) &= x_1 T(1,0) + x_2 T(0,1) \\
&= x_1 (a,b) + x_2 (c,d) \\
&= (ax_1 + cx_2, \; bx_1 + dx_2)
\end{aligned}
$$

3. (a). $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, is defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3,\ 2x_1 + x_2,\ -x_1 - 2x_2 + 2x_3)$$

A vector $(a, b, c) \in \mathbb{R}^3$ is in the range of T if the system of linear eq^ns,

$$\left.\begin{aligned} x_1 - x_2 + 2x_3 &= a \\ 2x_1 + x_2 &= b \\ -x_1 - 2x_2 + 2x_3 &= c \end{aligned}\right\}$$

has a solution. In other words, $(a,b,c) \in$ Range T if the ~~aug~~ row-reduced form of the augmented matrix is consistent. i.e

if the row-reduced form of $\begin{pmatrix} 1 & -1 & 2 & : a \\ 2 & 1 & 0 & : b \\ -1 & -2 & 2 & : c \end{pmatrix}$ is constitent.

After a sequence of ~~ent~~ elementary row-operation the above matrix reduces to,

$$(R : a) = \begin{pmatrix} 1 & 0 & 2/3 & : \frac{b+a}{3} \\ 0 & 1 & -4/3 & : \frac{b-2a}{3} \\ 0 & 0 & 0 & : c+b-a \end{pmatrix}$$

and clearly $(R : a)$ is consistent if $c + b - a = 0$.

Hence $(a, b, c) \in$ Range T if $b + c - a = 0$.

$\Rightarrow$ vectors in Range T are of the form $(b+c, b, c) = b(1,1,0) + c(1,0,1)$

$\Rightarrow$ Rank T $= 2$.

(b). If $(a,b,c) \in \mathbb{R}^3$ is in the null space of T (3)

then

$$T(a,b,c) = (0,0,0)$$

$$\Rightarrow \cancel{a+2} \Rightarrow (a-b+2c,\ 2a+b,\ -a-2b+2c) = (0,0,0)$$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Row-reducing the coef matrix we get

$$\begin{pmatrix} 1 & 0 & 2/3 \\ 0 & 1 & -4/3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} a + 2/3c &= 0 \\ b - 4/3c &= 0. \end{aligned}$$

$\Rightarrow$ $(a,b,c) \in$ Null space of T if $a = -\frac{2}{3}c$, $b = \frac{4}{3}c$

this implies that

$$\left(-\frac{2}{3}, \frac{4}{3}, 1\right)\mathbb{R} = \text{Null space of } T.$$

$\Rightarrow$ Nullity of $T = 1$.