MTH 101 - Symmetry
Assignment 10 & Notes

Defn : Let V and W be two vector spaces over $\mathbb{R}$. A **linear transformation** from V to W is a map $T : V \to W$ such that

i. $T(v_1 + v_2)= T(v_1) + T(v_2)$ for all $v_1, v_2 \in V$.

ii. $T(cv) = cT(v)$ for all $c \in \mathbb{R}$ and $v \in V$.

Notice that putting $c = 0$ in (ii), we get $T(0.v) = 0.T(v)$ for all $v \in V$. Since for any vector $v \in V$, $0.v$ equal to the $\mathbf{0}$ vector, therefore a linear transformation always maps the zero vector $\mathbf{0_V}$ of V to the zero vector $\mathbf{0_W}$ of W. Hence the **only linear transformations from the 1-dimensional vector space $\mathbb{R}|_{\mathbb{R}}$ to $\mathbb{R}|_{\mathbb{R}}$ are**

$$T(v) = cv, \quad \text{for some } c \in \mathbb{R}.$$

For $c = 0$, we get the zero transformation, $T(v) = 0$ for all $v \in \mathbb{R}$.
Given a basis $B_V = \{v_1, v_2, \cdots, v_n\}$ of an n-dimensional vector space $V|_{\mathbb{R}}$, we know that any vector $v \in V$ is of the form $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ for some $c_i \in \mathbb{R}$, i=1,2,..,n. Hence if $T : V \to W$ is a linear transformation, then to determine $T(v)$ it is sufficient to know the values of $\{T(v_i) : i = 1, 2 \cdots, n\}$.

Defn: For a linear transformation $T : V \to W$,

i. The **null space of** T, denoted by N_T or $N(T)$ is given as follows:

$$N(T) = \{v \in V : Tv = 0\}.$$

(Check that N_T is a subspace of V.) The **nullity** of T is defined as the **dimensional of** N_T.

ii. The **range of** T is defined as follows:

$$\text{Range}(T) = \{w \in W : Tv = w, \text{ for some } v \in V\}.$$

(Check that Range(T) is a subspace of W.) The **rank** of T is defined as the **dimensional of Range(T)**.

Defn: Given an ordered basis $B_V = \{v_1, \cdots, v_n\}$ of $V|_{\mathbb{R}}$, if $v \in V$ is of the form

$$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n,$$

then the **column representation of v with respect to the ordered basis B_V** is

$$[v]_{B_V} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}. \quad (1)$$

Defn: <u>Matrix of T relative to the ordered basis $[B_V : B_W]$</u>. Let $B_V = \{v_1, \cdots, v_n\}$ be an ordered basis of $V|_{\mathbb{R}}$, $B_W = \{w_1, \cdots, w_k\}$ be an ordered basis of $W|_{\mathbb{R}}$ and let $T : V \to W$ be a linear transformation. If for $v_j \in B_V$,

$$T(v_j) = a_{1j}w_1 + a_{2j}w_2 + \cdots + a_{kj}w_k,$$

then the **matrix of T relative to the ordered basis $[B_V : B_W]$**, is written as follows:

$$T_{[B_V:B_W]} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix}$$

Notice that the j^{th} *column of the matrix* $T_{[B_V,B_W]}$ *is a column representation of* $T(v_j)$ *with respect to the ordered basis* B_W of W and for $v \in V$ with $[v]_{B_V}$ given by (1),

$$[Tv]_{B_W} = T_{[B_V:B_W]}.[v]_{B_V},$$

where $T_{[B_V:B_W]}.[v]_{B_V}$ denotes the multiplication of the $k \times n$ matrix $T_{[B_V:B_W]}$, with the $k \times 1$ column matrix $[v]_{B_V}$.

Defn: Given a linear transformation $T : V \to V$, a non-zero vector $v \in V$ is said to be an **eigenvector** of T if there exists $c \in \mathbb{R}$ such that

$$T(v) = cv.$$

T is said to be **diagonalizable** if there exists a basis of V consisting of eigenvectors.
Notice that if $B_V = \{v_1, \cdots, v_n\}$ is an ordered basis of $V|_{\mathbb{R}}$ consisting of eigenvectors, i.e. if $T(v_i) = r_i v_i$, for $i = 1, 2 \cdots, n$ then

$$T_{[B_V, B_V]} = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ 0 & 0 & \vdots & \vdots \\ 0 & 0 & \cdots & r_n \end{pmatrix},$$

which is a diagonal matrix.

Defn: Let $B_1 = \{v_1, \cdots, v_n\}$ and $B_2 = \{w_1, \cdots, w_n\}$ be two ordered basis for $V|_{\mathbb{R}}$ and $C : V \to V$ be a linear transformation such that $B(v_i) = w_i$ for $i = 1, \cdots, n$. If $w_i = b_{1i}v_1 + b_{2i}v_2 + \cdots + b_{ni}v_n$, then notice that

$$C_{[B_1,B_2]} = I_n, \quad C_{[B_1,B_1]} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

Notice that the matrix $C_{[B_1,B_1]}$ is the transpose of the change of basis matrix relative to $[B_1, B_2]$.
• **In literature,$C_{[B_1,B_1]}$ is referred to as the change of basis matrix relative to $[B_1, B_2]$. Henceforth, we shall denote it by $c_{[B_1,B_2]}$ and refer to the matrix $c_{[B_1,B_2]}$ as the change of basis matrix relative to $[B_1, B_2]$.**

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map defined by $T(x_1, x_2) = (x_1, 0)$. Let B_1 be the standard ordered basis of $\mathbb{R}^2$ and let $B_2 = \{v_1 = (1, 1), v_2 = (-1, 2)\}$ be another ordered basis of $\mathbb{R}^2|_{\mathbb{R}}$.
   - What is the matrix of T relative to the ordered basis $[B_1, B_2]$.
   - What is the matrix of T relative to the ordered basis $[B_2, B_1]$.
   - What is the matrix of T relative to the ordered basis $[B_2, B_2]$.
   - What is the matrix of T relative to the ordered basis $[B_3, B_3]$ where B_3 is the ordered basis $B_3 = \{v_2, v_1\}$.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map defined by $T(x_1, x_2) = (-x_2, x_1, x_1 + x_2)$. Let B_1 be the standard ordered basis of $\mathbb{R}^2$, B_2 be the standard ordered basis of $\mathbb{R}^3$ and let $B = \{v_1 = (1, 1, 1), v_2 = (-1, 2, 0), v_3 = (1, 0, 1)\}$ be another ordered basis of $\mathbb{R}^3|_{\mathbb{R}}$.
   - What is the matrix of T relative to the ordered basis $[B_1, B_2]$.
   - What is the matrix of T relative to the ordered basis $[B_1, B]$.
   - Is there a relation between the matrices $T_{[B_1,B_2]}$ and $T_{[B_1,B_2]}$.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $Tv = A[v]_{\{e_1,e_2,e_3\}}$ where $\{e_1, e_2, e_3\}$ denotes the standard ordered basis of $\mathbb{R}^3$. Determine the rank and nullity of T.

4. Determine the eigenvectors of the following linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, whenever they exist.
   i. $T(x_1, x_2) = (x_1 - x_2, x_2)$.
   ii. $T(x_1, x_2) = (2x_1 + x_2, 2x_1 - x_2)$.