Assignment 10

1. $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined by $T(x_1, x_2) = (x_1, 0)$

Let $B_1 = \{e_1 = (1,0),\ e_2 = (0,1)\}$

$B_2 = \{v_1 = (1,1),\ v_2 = (-1,2)\}$

$(1,0) = a_1(1,1) + a_2(-1,2) = (a_1 - a_2,\ a_1 + 2a_2)$

$\Rightarrow a_1 = -2a_2,\ 1 = -2a_2 - a_2 \Rightarrow a_2 = -\frac{1}{3}$

$\Rightarrow (1,0) = \frac{2}{3} v_1 - \frac{1}{3} v_2.$

$T(1,0) = (1,0) = \frac{2}{3} v_1 - \frac{1}{3} v_2.$

$T(0,1) = (0,0) = 0 \cdot v_1 + 0 \cdot v_2.$

- $[T]_{[B_1, B_2]} = \left([Te_1]_{B_2}\ [Te_2]_{B_2} \right) = \begin{pmatrix} 2/3 & 0 \\ -1/3 & 0 \end{pmatrix}$

- $[T]_{[B_2, B_1]} = \left([Tv_1]_{B_1}\ [Tv_2]_{B_1} \right)$

$Tv_1 = T(1,1) = (1,0) = e_1$

$Tv_2 = T(-1,2) = (-1,0) = -e_1$

$\Rightarrow [T]_{[B_2, B_1]} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

- $[T]_{[B_2, B_2]} = \left([Tv_1]_{B_2}, [Tv_2]_{B_2} \right)$

$Tv_1 = T(1,1) = (1, 0) = \frac{2}{3} v_1 - \frac{1}{3} v_2$

$Tv_2 = T(-1,2) = (-1,0) = -\frac{2}{3} v_1 + \frac{1}{3} v_2$

$$\Rightarrow [T]_{[B_2, B_2]} = \begin{pmatrix} 2/3 & -2/3 \\ -1/3 & 1/3 \end{pmatrix}$$

- $B_3 = \{ \overset{w_1}{\overset{\shortparallel}{v_2}}, \overset{w_2}{\overset{\shortparallel}{v_1}} \}$

$Tw_1 = Tv_2 = -\frac{2}{3} w_2 + \frac{1}{3} w_1$

$Tw_2 = Tv_1 = \frac{2}{3} v_1 - \frac{1}{3} v_2 = \frac{2}{3} w_2 - \frac{1}{3} w_1$

$$[T]_{[B_3, B_3]} = \left([Tw_1]_{B_3}, [Tw_2]_{B_3} \right) = \begin{pmatrix} 1/3 & -1/3 \\ -2/3 & 2/3 \end{pmatrix}$$

2. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is the linear map defined by

$T(x_1, x_2) = (-x_2, x_1, x_1 + x_2)$

$B_1 = \{ e_1 = (1, 0), e_2 = (0,1) \}$

$B_2 = \{ \bar{e}_1 = (1,0,0), \bar{e}_2 = (0,1,0), \bar{e}_3 = (0,0,1) \}$

$B = \{ v_1 = (1,1,1), v_2 = (-1,2,0), v_3 = (1,0,1) \}$

$Te_1 = (0,1,1) \qquad Te_2 = (-1,0,1)$

- $[T]_{[B_1, B_2]} = \left([Te_1]_{B_2} \; [Te_2]_{B_2} \right)$

Note $Te_1 = (0,1,1) = 0.\bar{e}_1 + \bar{e}_2 + \bar{e}_3$

$Te_2 = (-1,0,1) = -\bar{e}_1 + 0.\bar{e}_2 + \bar{e}_3$

$$\Rightarrow [T]_{[B_1,B_2]} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

- $[T]_{[B, B]} = \left([Te_1]_B \; [Te_2]_B \right)$

$$Te_1 = (0,1,1) = a_1 v_1 + a_2 v_2 + a_3 v_3$$
$$= a_1(1,1,1) + a_2(-1,2,0) + a_3(1,0,1)$$

$$\Rightarrow \left.\begin{aligned} a_1 - a_2 + a_3 &= 0 \\ 2a_2 + a_1 &= 1 \\ a_1 + a_3 &= 1 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} a_3 &= 1 - a_1 \\ a_2 &= \frac{1-a_1}{2} \end{aligned}$$

$$a_1 - \frac{(1-a_1)}{2} + (1-a_1) = 0$$

$$\Rightarrow 2a_1 + 1 - a_1 = 0$$

$$\Rightarrow a_1 = -1.$$

$$\Rightarrow a_2 = \frac{1+1}{2} = 1$$

$$a_3 = 2$$

$$\Rightarrow Te_1 = -1.v_1 + v_2 + 2.v_3.$$

$$Te_2 = (-1,0,1) = b_1 v_1 + b_2 v_2 + b_3 v_3$$
$$= b_1(1,1,1) + b_2(-1,2,0) + b_3(1,0,1)$$

$$\Rightarrow \left.\begin{aligned} -1 &= b_1 - b_2 + b_3 \\ 0 &= b_1 + 2b_2 \\ 1 &= b_1 + b_3 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} b_2 &= -b_1/2 \\ b_3 &= 1 - b_1 \end{aligned}\right\} \begin{aligned} -1 &= \cancel{b_1} + \frac{b_1}{2} + 1 - \cancel{b_1} \\ &\Rightarrow -2.2 = b_1 \end{aligned}$$

$$\Rightarrow \quad Te_2 = -4v_1 + 2v_2 + 5v_3$$

$$\Rightarrow \quad [T]_{[B,B]} = \left([Te_1]_B \ [Te_2]_B \right)$$

$$= \begin{pmatrix} -1 & -4 \\ 1 & 2 \\ 2 & 5 \end{pmatrix}$$

- Note $[T]_{[B_1,B]} = \left([Te_1]_B \ [Te_2]_B \right)$.

$$[Te_1]_B = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$\Rightarrow Te_1 = a_1 v_1 + a_2 v_2 + a_3 v_3.$$

$$[Te_1]_{B_2} = a_1 [v_1]_{B_2} + a_2 [v_2]_{B_2} + a_3 [v_3]_{B_2}$$

$$\therefore \left([v_1]_{B_2} \ [v_2]_{B_2} \ [v_3]_{B_2} \right) [T]_{[B_1,B]} = [T]_{[B_1,B_2]}$$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -4 \\ 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

$$\therefore \boxed{\underline{[T]_{[B_1,B_2]}} = \underline{\left([v_1]_{B_2} \ [v_2]_{B_2} \ [v_3]_{B_2} \right) [T]_{[B_1,B]}}}$$

where $\{v_1, v_2, v_3\}$ is an ordered basis of B.

3. $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is defined by $Tv = Av$ where A is a 3×3 matrix and the vector v is written as a column vector wrt the standard ordered basis $\{e_1, e_2, e_3\}$.

Then Nullity of T = dimension of N_T

where

$$N_T = \{ [v]_{[e_1,e_2,e_3]} \in \mathbb{R}^3 : A[v]_{[e_1,e_2,e_3]} = 0 \}$$

i.e given such a question determine the spanning set for the space $AX = 0$, and determine $\dim N_T$.

and the rank $T = 3 = \dim \mathbb{R}^3 -$ nullity of T by rank-nullity theorem.

4(i) $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$T(x_1, x_2) = (x_1 - x_2, x_2)$.

If (x_1, x_2) is an eigenvector of T, then $\exists\, c \in \mathbb{R}$ st $T(x_1, x_2) = c(x_1, x_2)$

$\Rightarrow (x_1 - x_2, x_2) = (cx_1, cx_2)$ —— ⊛

equating the components of ⊛ we get

$$\left.\begin{array}{l} x_1 - x_2 = cx_1 \\ x_2 = cx_2 \end{array}\right\} \Rightarrow \begin{cases} (1-c)x_1 = x_2 \\ (1-c)x_2 = 0 \end{cases}$$

Substituting the value of x_2 in the second eqn, we get

$$(1-c)^2 x_1 = 0.$$

if $x_1 \neq 0$, $(1-c)^2 = 0 \Rightarrow c = 1$.

$\therefore$ $c = 1$ is an eigenvalue of T

Note that if $(x_1 - x_2, x_2) = (x_1, x_2)$

then $x_1 - x_2 = x_1 \Rightarrow x_2 = 0$

$\therefore$ $(x_1, 0)$ is an eigenvector corresponding to eigenvalue $c=1$,

and it is easy to see that

$T(1,0) = (1,0)$ $\therefore$ $(1,0)$ is infact an eigenvector of T.

(ii) $T(x_1, x_2) = (2x_1 + x_2, 2x_1 - x_2)$.

If (x_1, x_2) is an eigenvector of T, then $\exists$ c st $T(x_1, x_2) = c(x_1, x_2)$

$$\Rightarrow (2x_1 + x_2, 2x_1 - x_2) = (cx_1, cx_2)$$

$$\Rightarrow \left.\begin{array}{l} (2-c)(x_1) + x_2 = 0 \\ 2x_1 - (1+c)x_2 = 0 \end{array}\right\} \Rightarrow x_1 = \frac{(1+c)}{2} x_2 \text{ and } (2-c)\frac{(1+c)}{2} x_2 + x_2 = 0$$

$$\Rightarrow \left((2-c)(1+c)+2\right)x_2 = 0.$$

$$\Rightarrow (2+2c-c-c^2+2)\,x_2 = 0.$$

$\because$ $x_2 = 0$ implies $x_1 = 0$ $\therefore$ $x_2 \neq 0$ (as a 0 vector cannot be an eigenvector)

$$\Rightarrow 2+2c-c-c^2+2 = 0$$

$$\Rightarrow c^2 - c - 4 = 0$$

$$\Rightarrow c = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-4)}}{2.1}$$

(Using the formula $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ are the roots of the eqn $ax^2+bx+c=0$)

$$= \frac{1 \pm \sqrt{1+16}}{2}$$

$$= \frac{1 \pm \sqrt{17}}{2} \in \mathbb{R}.$$

$\therefore$ $T:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $(2x_1+x_2, 2x_1-x_2)$ has two distinct eigenvalues, namely $\frac{1+\sqrt{17}}{2}$ and $\frac{1-\sqrt{17}}{2}$.

Using the relation $x_1 = \frac{(1+c)}{2}x_2$ and putting $x_2 = 1$ and the value of ~~the~~ c ~~eig~~ we see that

$$\left(\frac{1+\left(\frac{1+\sqrt{17}}{2}\right)}{2},\ 1\right)$$

is ~~the~~ a eigenvector corresponding to eigenvalue $\frac{1+\sqrt{17}}{2}$

and $\left(\frac{1+\left(\frac{1-\sqrt{17}}{2}\right)}{2}, 1 \right)$ is an eigenvector

corresponding to eigenvalue $\frac{1-\sqrt{17}}{2}$.