Philosophy of Science: Conditional and Indirect Proof

Reference: Symbolic Logic (I. M. Copi)

1 The Rule of Conditional Proof

In this lecture, we study arguments whose conclusions are conditional statements. Clearly, to every argument there corresponds a conditional statement whose antecedent is the conjunction of the arguments premisses and whose consequent is the arguments conclusion. **An argument is valid if and only if its corresponding conditional is a tautology**.

Suppose an argument has a conditional statement of the form $A \implies B$ as its conclusion and suppose we denote the conjunction of its premisses by P, then from above we know that the argument is valid if and only if the conditional $P \implies (A \implies B)$ is a tautology.

The rule of of conditional proof says that we can infer the validity of an argument of the form

$$\begin{array}{l} P \\ \therefore \ A \implies B \end{array}$$

from the formal proof of validity of the argument

$$\begin{array}{l} P \\ A \\ \therefore \ B \end{array}$$

That is one may think of the antecedent in the conclusion as an additional premiss and then deduce the consequent of the conclusion by a sequence of elementary valid argument.

The rule can be applied multiple times as seen in the example below.

Example 1.1. *Consider the argument:*

$$\begin{array}{c} A \implies (B \implies C) \\ B \implies (C \implies D) \\ \therefore A \implies (B \implies D) \end{array}$$

By the rule of conditional proof, proving the validity of the above argument is same as proving the validity of the following argument:

$$\begin{aligned}&A \implies (B \implies C)\\&B \implies (C \implies D)\\&A\\\therefore\,&(B \implies D)\end{aligned}$$

This can further be reduced to

$$\begin{aligned}&A \implies (B \implies C)\\&B \implies (C \implies D)\\&A\\&B\\&\therefore D\end{aligned}$$

Example 1.2. *Consider the argument:*

$$\begin{aligned}&(A \vee B) \implies (C \cdot D)\\&(D \vee E) \implies F\\&\therefore \quad A \implies F\end{aligned}$$

This may be written as:

1. $(A \vee B) \implies (C \cdot D)$
2. $(D \vee E) \implies F \qquad (\therefore A \implies F)$
3. $A. \qquad (\therefore F)$
4. $A \vee B \qquad (Add.\ 3)$
5. $C \cdot D \qquad (M.P.\ 1, 4)$
6. $D \cdot C \qquad (Com.\ 5)$
7. $D \qquad (Simp.\ 6)$
8. $D \vee E \qquad (Add.\ 7)$
9. $F \qquad (M.P.\ 2, 8)$

where, the first two lines follow from the above discussed rule of conditional proof.

2 Reductio Ad Absurdum or The Rule of Indirect Proof

The idea is as follows: We start by assuming the opposite of the conclusion (that we want to prove from the premisses). If that assumption leads to a contradiction, or 'reduces to an absurdity', then the assumption must be false, and so its negation (which was the original statement of the conclusion) must be true. That is, **An indirect proof of validity for a given argument is constructed by assuming, as an additional premiss, the negation of its conclusion, and then deriving an explicit contradiction from the augmented set of premisses.**

Example 2.1. *Consider the following argument.*

$$\begin{aligned} & A \implies (B \cdot C) \\ & (B \vee D) \implies E \\ & D \vee A \\ & \therefore \quad E \end{aligned}$$

We now give an indirect proof of validity of this argument.

1. $A \implies (B \cdot C)$
2. $(B \vee D) \implies E$
3. $D \vee A \qquad (\therefore E)$
4. $\sim E \qquad$ (*adding the negation - Indirect Proof (I.P.)*)
5. $\sim (B \vee D) \qquad (M.T.\ 2, 4)$
6. $\sim B \cdot \sim D$
7. $\sim D \cdot \sim B$
8. $\sim D$
9. A
10. $B \cdot C$
11. $B \qquad (Simp.\ 10)$
12. $\sim B \qquad (Simp.\ 6)$
13. $B \cdot \sim B$

the last line is a contradiction. This completes the indirect proof of the validity of the argument given above.

Exercise 2.2. *Fill in the details of the proof in example 3.1 by specifying the elementary valid arguments used to deduce steps 6, 7, 8, 9, 10 and 13.*

Exercise 2.3. *Construct both a formal proof of validity and an indirect proof for the following argument:*

$$\begin{aligned}
&(H \implies I) \cdot (J \implies K) \\
&(I \vee K) \implies L \\
&\sim L \\
&\therefore \quad \sim (H \vee J)
\end{aligned}$$