6

Deductive Systems

6.1 Definition and Deduction

In the preceding chapters a number of principles of logic have been set forth. These principles embody some knowledge about logic, but they do not constitute a *science* of logic, for *science* is *organized* knowledge. No mere list or catalog of truths is ever said to constitute a system of knowledge or a science. We have scientific knowledge only when the propositions setting forth what we know are organized in a systematic way, to display their interrelations. If a system of logic or a science of logical principles is to be achieved, those principles must be arranged or organized in a systematic fashion. This task will be attempted, on a limited scale, in the following chapters. But first it will be of interest to consider the general questions of what interrelations are important, and how propositions may be organized into a system or science.

All knowledge that we possess can be formulated in propositions, and these propositions consist of terms. In any science, some propositions can be deduced from or proved on the basis of other propositions. For example, Galileo's laws of falling bodies and Kepler's laws of planetary motion are all derivable from Newton's more general laws of gravitation and motion, and the discovery of these deductive interrelationships was an exciting phase in the development of the *science* of physics. Thus one important relationship among the propositions of a science is deducibility. Propositions that embody knowledge about a subject become a *science* of that subject when they are arranged or ordered by displaying some of them as conclusions deduced from others.

In any science, some of the terms involved in its propositions can be defined on the basis of other terms. For example, in physics again, *density* is defined as *mass per unit volume*, *acceleration* is defined as the *time rate of change in velocity*, and *velocity* is in turn defined as the *time rate of change of position*. This definition of some terms by means of others also serves to reveal interrelations among the propositions. It shows their concern with a common subject matter, and integrates the concepts of the science just as deductions integrate its laws or statements. Propositions that embody knowledge are helped to become a *science* when some of the words or symbols they contain are defined in terms of their other symbols.

The recognition of definition and deduction as important to science may suggest a specious ideal for scientific systems. It may be imagined that in an ideal science *all* propositions should be *proved*, by deducing them from others, and *all* terms should be *defined*. But this would be 'ideal' only in the sense of being impossible to realize. Terms can be defined only by means of other terms, whose meanings are presupposed and must be antecedently understood if the definitions are to explain the meanings of the terms being defined. And deductions can establish their conclusions only on the basis of premisses, which must already have been verified if the conclusions are really to be established by the proofs. Hence, if all terms or symbols of a system are to be defined *within the system*, there must be either infinite sequences of definitions, or circular definitions, as in a pocket dictionary which defines the word 'big' as meaning *large*, and the word 'large' as meaning *big*. Circular definitions are obviously worthless as explanations, and infinite sequences of definitions are worthless also, for no term will really be explained until the end is reached, and an infinite sequence has no end. Similarly, to prove *all* propositions there must be either infinite regressions of proofs or circular proofs. And these are equally objectionable.

It must be admitted that *within* a system of propositions which constitutes a science, not all propositions can be proved, and not all terms can be defined. It is not that there is some particular proposition that cannot be proved, or some particular term that cannot be defined, but rather that they cannot *all* be proved or defined without a vicious regression or circularity arising. The ideal of science, then, cannot be a system in which *every* proposition is proved and *every* term defined, but is rather one in which a minimum number of propositions suffice for the deduction of all the rest, and a minimum number of terms suffice for the definition of all the others. This ideal of knowledge is described as a *deductive system*.

6.2 Euclidean Geometry

Euclidean Geometry is the oldest example of systematized knowledge or science. Of historical interest and importance in its own right, it has the advantage (for our purpose) of being an example with which the reader has already come into contact in high school.

It is generally recognized that geometry, as a science, was originated and developed by the Greeks. Among the most important contributors to its development were the mathematicians Pythagoras and Euclid. And yet, geometrical truths were known to the Egyptians thousands of years earlier, as attested by their pyramids, already ancient in the time of Pythagoras (6th century B.C.). Records reveal that the Babylonians, even earlier, were familiar with various principles of geometry. If geometrical knowledge already existed before their time, in what sense did the Greeks originate the science of geometry? The answer has already been indicated. Before Pythagoras, man's geometrical knowledge consisted of a collection or catalog of almost wholly isolated facts. Geometrical truths constituted a mere list of useful empirical rules-of-thumb for surveying land or constructing bridges or buildings, and there was no system to their knowledge of geometrical truths. By introducing order into the subject, the Greeks transformed it from a mere body of isolated bits of knowledge into a science.

System was introduced into geometry by the deduction of some of its propositions from others. The propositions of geometry were ordered by listing earlier those which could be used as premisses in the demonstrations of those which were put later. This systematization of geometry was begun by Pythagoras and continued by his successors. It culminated in the *Elements* of Euclid (c.300 B.C.), in which all geometrical propositions were arranged in order, beginning with Axioms, Definitions, and Postulates, and continuing with Theorems deduced from the initial propositions. Geometry was cast by the Greeks into the form of a deductive system. Theirs was the first deductive system ever devised, and so great was the achievement that it has served as a model for scientific thought down to the present time. Even today the most advanced sciences are those which most nearly approximate the form of a deductive system. These are the sciences which have achieved a relatively small number of very general principles from which a relatively large number of other laws and special cases may be derived. Parts of physics have actually been formulated as deductive systems, and similar attempts have been made, with somewhat less impressive results, in parts of biology and psychology also. Perhaps the boldest attempt in this direction was that of Spinoza, whose most important work, the *Ethics*, was written in 'geometrical' form. Starting with axioms and definitions, Spinoza attempted to deduce the rest of his metaphysical and ethical doctrines as theorems provable on the basis of those initial assumptions.

Euclid begins his geometry with definitions of some of the terms used in its development. Thus Definition 1 reads: 'A point is that which has no parts', and Definition 2 reads: 'A line is length without breadth'.[1] Euclid does not attempt to define *all* his terms, of course. The first two definitions define the terms 'point' and 'line' respectively. The words *used* in these definitions, such as 'parts', 'length', and 'breadth' are not themselves defined but are among the *undefined terms* of the system for Euclid. As more new terms are introduced their definitions make use of previously defined terms as well as the original undefined ones. Thus Definition 4: 'A straight line is . . . [a line] . . . which lies evenly between its extreme points', makes use not only of such undefined terms as 'evenly' and 'between', but also the previously defined terms 'point' and 'line'.

The use of *defined* terms is, from the point of view of logic, a matter of convenience only. Theoretically, every proposition that contains defined terms

[1] These and the following are quoted from the Todhunter edition of *The Elements of Euclid*, No. 891 of Everyman's Library, London and New York.

can be translated into one that contains only undefined ones by replacing each occurrence of a defined term by the sequence of undefined terms which was used to define it. For example, Postulate 1: 'Let it be granted that a straight line may be drawn from any one point to any other point', which contains the defined terms 'straight', 'line', and 'point', can be expressed without using those defined terms as: 'Let it be granted that a length without breadth which lies evenly between its extreme parts which (themselves) have no parts may be drawn from any one thing which has no parts to any other thing which has no parts'. But this version of the Postulate is extremely awkward. Although they are theoretically eliminable, in actual practice a considerable economy of space, time, and effort is effected by using relatively brief defined terms to replace lengthy sequences or combinations of undefined ones.

In setting up his deductive system of geometry, Euclid divided his unproved propositions into two groups, one called 'Axioms', the other called 'Postulates'. He gave, however, no reason for making this division, and there seems to be no very clear basis for distinguishing between them. Possibly he felt that some were more *general* than others, or psychologically more *obvious*. The contemporary practice is to draw no such distinction, but to regard all the unproved, initial propositions of a deductive system as having the same standing, and to refer to them all, indifferently, as 'axioms' or as 'postulates', without attaching any difference in meaning to those two terms.

Every deductive system, on pain of falling into circularity or a vicious regression, must contain some axioms (or postulates) which are assumed but not proved within the system. They need not be *precarious* assumptions, or *mere* assumptions. They may be very carefully and convincingly established—but they are *not proved within the system itself*. Any argument intended to establish the truth of the axioms is definitely *outside* the system, or *extra-systematic*.

The older conception of Euclidean geometry held not only that all of its theorems followed logically from its axioms, and were therefore just as *true* as the axioms, but also that the axioms were *self-evident*. It is in this tradition to regard any statement as 'axiomatic' when its truth is beyond all doubt, being evident in itself and not requiring any proof. It should be clear from what has already been said, however, that we are *not* using the word 'axiom' in *that* sense. No claim is made that the axioms of any system are self-evidently true. Any proposition of a deductive system is an axiom of that system if it is assumed rather than proved in that system. This modern point of view has arisen largely as a consequence of the historical development of geometry and physics.

The self-evident truth of the Euclidean axioms (and postulates) was long believed. It was not believed quite whole-heartedly, however. Most of the axioms, such as Axiom 9: 'The whole is greater than its part', were not questioned; but while there was no doubt about the *truth* of Axiom 12 (the famous 'parallel Postulate'), there was considerable scepticism about its 'self-evidence'. Axiom 12 reads: 'If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles'.[2] Proclus, a fifth century A.D. commentator, wrote of it: 'This ought even to be struck out of the Postulates altogether; for it is a theorem involving many difficulties . . .'.[3] That is, although its *truth* was not questioned, its *self-evidence* was denied, which was deemed sufficient reason to relegate it from its exalted position as axiom to the less exalted status of a mere theorem.

The history of mathematics is filled with attempts to prove the proposition in question as a theorem, either by deducing it from the remaining axioms of Euclid, or from those axioms supplemented by some more nearly 'self-evident' additional assumption. The latter kind of attempt was pretty uniformly unsuccessful, because every additional or alternative assumption strong enough to permit the deduction of the parallel postulate turned out to be no more self-evident than Euclid's own hypothesis. The first kind of attempt failed also; it was just not possible to deduce the parallel postulate from the others. The most fruitful attempt was that of the Italian mathematician Gerolamo Saccheri (1667–1733), who *replaced* the parallel postulate by alternative, contrary assumptions, and then sought to derive a contradiction from them together with Euclid's other axioms. Had he succeeded in doing so, he would have obtained a *reductio ad absurdum* proof of the parallel postulate. He derived many theorems that he regarded as *absurd* because they were so different from common sense or ordinary geometrical intuition. He believed himself to have succeeded thus in demonstrating the parallel postulate, and in 'vindicating Euclid'. But his derived theorems, while 'absurd' in the sense of violating ordinary geometrical intuitions, were *not* 'absurd' in the logical or mathematical sense of being self-contradictory. Instead of proving the parallel postulate, Saccheri (unknowingly) did something more important: he was the first to set up and develop a system of non-Euclidean geometry.

The parallel postulate is in fact *independent* of the other Euclidean postulates—although it was not *proved* to be so until the modern period. It is independent of the other postulates in the sense that neither it nor its denial is deducible from them. Alternative systems of 'geometry', non-Euclidean geometries, were subsequently developed, notably by Lobachevsky and Riemann. These were long regarded as ingenious fictions, mere mathematical playthings, in contrast with the Euclidean geometry which was 'true' of the real space about us. But subsequent physical and astronomical research along lines suggested by Einstein's theory of relativity has tended to show that—to the extent that the question is significant—'real' or physical space is more probably non-Euclidean than Euclidean. In any event, the truth or falsehood of its axioms is a purely *external* property of any deductive system. The truth

[2]Listed as Postulate 5 by Sir Thomas L. Heath, in *The Thirteen Books of Euclid's Elements*, Cambridge, Eng., Cambridge University Press, 1926. For an interesting discussion of the history of the parallel postulate, the reader is referred to pages 202 ff. of Volume I of that work.

[3]Ibid., p. 202.

of its propositions is an extrasystematic consideration. It is no doubt important to the extent that a deductive system is ordered *knowledge;* but when we concentrate our attention on the system as such, its *order* is its more important characteristic.

From the purely mathematical or logical point of view, a deductive system can be regarded as a vast and complex argument. Its premisses are the axioms, and its conclusion is the conjunction of all the theorems deduced. As with any other argument, the logical question does not concern the truth or falsehood of the premisses, but the validity of the inference. Granted the truth of the axioms, does the truth of the theorems necessarily follow? That is the question with which the logician and the mathematician are concerned. The answer is, of course, yes—*if* the demonstrations of the theorems are all valid arguments. Hence the most important aspect of any deductive system is the cogency with which its theorems are proved. In the rigorous development of deductive systems in abstraction from the extrasystematic explanation of their undefined terms, the question of truth or falsehood is obviously irrelevant.

6.3 Formal Deductive Systems

There are serious errors in the system of geometry set forth by Euclid in his *Elements*. Indeed, a mistake occurs in his very first proof. The flaw in his proof, paradoxically enough, was the result of his knowing too much about his subject. He did not appeal to his explicitly stated axioms alone as premisses, but depended also upon what might be called his geometrical intuition.[4] Where a chain of argument involves familiar notions, there is always the danger of assuming more than the explicitly stated premisses warrant. That is particularly serious in the development of a deductive system, for any attempted systematization which appeals to new and unacknowledged assumptions in the derivations of its theorems thereby *fails* to achieve its aim. In a deductive system the theorems must be deduced *rigorously* from the stated postulates. If they are not, however true they may be, the result falls short of the goal of systematization.

Since lapses from rigor are most often occasioned by too great familiarity with the subject matter, mathematicians have found it helpful to minimize or eliminate such familiarity in the interest of achieving greater rigor. In the case of geometry, that end is accomplished by abstracting from the meanings of such geometrical words as 'point', 'line', and 'plane', and developing the theorems as purely formal consequences of the postulates. The familiar geometrical words, with all their associations and suggestions, are replaced by

[4] Euclid's proof and a brief discussion of his mistake can be found on pages 241–243 of Volume I of *The Thirteen Books of Euclid's Elements*, by Heath, op. cit. An example of how the same type of error can lead to conclusions that are false or even self-contradictory can be found on pages 77–78 of *Mathematical Recreations and Essays*, by W. W. Rouse Ball, New York, The Macmillan Company, 1940.

arbitrary symbols. Instead of deductive systems explicitly and avowedly concerned with geometrical entities, mathematicians today develop *formal* deductive systems whose primitive or undefined terms include arbitrary, uninterpreted symbols, usually letters of the Greek or Latin alphabets. Since the undefined terms of a *formal* deductive system include arbitrary symbols, its postulates are not propositions at all, but mere formulas, and so are the theorems.

Deductive relationships can exist, of course, among mere formulas as well as among propositions. Thus the formula 'all *F*'s are *H*'s' is logically deducible from the formulas 'all *F*'s are *G*'s' and 'all *G*'s are *H*'s'. Because the postulates and theorems of a formal deductive system are formulas rather than propositions, the proofs of theorems can proceed unhampered by familiar associations and unconscious assumptions. Moreover, because the formulas are not propositions, the question of their truth is strictly irrelevant and does not arise.

More than rigor is gained by the formal development of deductive systems. Since some of the symbols of a formal deductive system are arbitrary uninterpreted symbols, it may be possible to give them different, alternative interpretations. And since the theorems are formal consequences of the axioms, any interpretation of the arbitrary symbols which makes the axioms true will necessarily make the theorems true also. The additional advantage of generality is thus gained. An example may help to make this clear. Given some knowledge about astronomy, it may be desired to set up a deductive system for that subject. To avoid the errors into which familiarity with the subject matter may lead in deducing theorems from the axioms chosen, the system may be developed *formally*. Instead of taking, say, 'stars' and 'planets' among the undefined terms, one may take '*A*'s and '*B*'s. The axioms and theorems will contain these symbols, and when the system is developed, all its formulas may be interpreted by letting the symbol '*A*' designate stars and the symbol '*B*' designate planets. Now, if the axioms are *true* when so interpreted, the theorems must be true also, and the formal system with this interpretation will constitute a science or deductive system of astronomy. But it may be possible to find a *different* interpretation of the symbols '*A*' and '*B*' which also makes the axioms true (and hence the theorems also). The formulas of the system might be made into different but equally true statements by letting the symbol '*A*' designate atomic nuclei and the symbol '*B*' designate electrons. Could this be done (and at one stage in the history of atomic physics it seemed highly plausible), the original formal system with this second interpretation would constitute a science or deductive system of atomic physics. Hence developing a deductive system formally, *i.e.*, not interpreting its undefined terms until after its theorems have all been derived, not only helps achieve rigor in its development, but also achieves greater generality because of the possibility of finding alternative interpretations for it (and applications of it). This kind of advantage is often realized in pure mathematics. For example, different interpretations of its arbitrary primitive symbols will transform the same formal deductive system into the theory of real numbers, on the one hand, or into the theory of points on a straight line, on the other. That fact provides the theoretical foundation for the branch of mathematics called *Analytical Geometry.*

As the term is being used here, a *formal deductive system* is simply a deductive system, consisting of axioms and theorems, some of whose undefined or primitive terms are arbitrary symbols whose interpretation is completely extrasystematic. In addition to those special undefined terms, and others defined by means of them, the formulas (axioms and theorems) of the system contain only such logical terms as 'if . . . then . . .', 'and', 'or', 'not', 'all', 'are', and the like, and possibly (unless the system is intended for arithmetic itself) such arithmetical terms as 'sum' and 'product', and numerical symbols.

6.4 Attributes of Formal Deductive Systems

Usually, though not always, a formal deductive system is set up with some particular interpretation 'in mind'. That is, the investigator has some knowledge about a certain subject, and wishes to set up a system adequate for its expression. When the formal system has been constructed, the question naturally arises as to whether or not it is adequate to the formulation of all the propositions it is intended to express. If it is, it may be said to be 'expressively complete' *with respect to that subject matter.* We are here discussing what can be *said* in the system, *not* what can be proved. With respect to a given subject matter, a formal deductive system is 'expressively complete' when it is possible to assign meanings to its undefined terms in such a way that every proposition about that subject matter can be *expressed* as a formula of the system. Whether the *true* propositions can be *proved as theorems* or not is another question, which will be discussed below.

A system is said to be *inconsistent* if two formulas, one of which is the denial or contradictory of the other, can both be proved as theorems within it. A system is *consistent* if it contains no formula such that both the formula and its negation are provable as theorems within it. As was shown in Chapter 3, a contradiction logically entails any proposition whatever. Hence a derivative definition or criterion for consistency can be formulated as follows: Any system is consistent if it contains (that is, can express) a formula that is not provable as a theorem within it. This is known as the 'Post criterion for consistency', having been enunciated by the American mathematician and logician, E. L. Post. Consistency is of fundamental importance. An inconsistent deductive system is worthless, for all of its formulas are provable as theorems, including those which are explicit negations of others. When the undefined terms are assigned meanings, these contradictory formulas become contradictory propositions, and cannot possibly all be true. And since they cannot possibly be true, they cannot serve as a systematization of knowledge—for knowledge is expressed in true propositions only.

If one succeeds in deriving both a formula and its negation as theorems of a system, that proves the system inconsistent. But if one tries and does not succeed in deriving both a formula and its negation as theorems, that does *not* prove the system to be consistent, for it may only reflect a lack of ingenuity at making proofs on the part of the investigator. How then can the consistency of a deductive system be established? One method of proving the consistency of a formal deductive system is to find an interpretation of it in which all its axioms and theorems are true propositions. Since its theorems are logical consequences of its axioms, any interpretation which makes its axioms true will make its theorems true also. Hence it is sufficient for the purpose of proving a system consistent to find an interpretation which makes all of its axioms true.

The axioms of a deductive system are said to be *independent* (or to exhibit *independence*) if no one of them can be derived as a theorem from the others. A deductive system which is not consistent is logically objectionable and utterly worthless, but there is no *logical* objection to a deductive system whose axioms are not independent. However, it is often felt that making more assumptions than necessary for the development of a system is extravagant and inelegant, and should be avoided. When a formula need not be assumed as an axiom, but can be proved as a theorem, it *ought* to be proved and not assumed, for the sake of 'economy'. A set of axioms which are not independent is said to be 'redundant'. A redundant set of axioms is aesthetically inelegant, but it is not logically 'bad'.

If one of the axioms of a system *can* be derived from the remaining ones, the set of axioms is thereby shown to be redundant. But if one tries and is not able to derive any of the axioms from the remaining ones, they are *not* thereby shown to be independent, for the failure to find a demonstration may be due only to the investigator's lack of ingenuity. To prove any particular axiom independent of the others, it suffices to find an interpretation which makes the axiom in question *false* and the remaining ones all *true*. Such an interpretation will prove that the axiom in question is not derivable as a theorem from the others, for if it were, it would be made true by any assignment of meanings which made the others true. If such an interpretation can be found for each axiom, this will prove the set of axioms to be independent.

The notion of *deductive completeness* is a very important one. The term 'completeness' is used in various senses. In the least precise sense of the term we can say that a deductive system is complete if all the *desired* formulas can be proved within it. We may have an extrasystematic criterion for the truth of propositions about the subject matter for which we constructed the deductive system. If we have, then we may call that system complete when all of its formulas which become true propositions on the intended interpretation are provable formulas or theorems of the system. (In any sense of the term, an *inconsistent* system will be *complete*, but in view of the worthlessness of inconsistent systems, we shall confine our attention here to *consistent* systems.)

There is another conception of *completeness* which can be explained as follows. Any formal deductive system will have a certain collection of special undefined or primitive terms. Since any terms definable within the system are theoretically eliminable, being replaceable in any formula in which they occur by the sequence of undefined terms by means of which they were defined, we shall ignore defined terms for the present. All formulas which contain no terms other than these special undefined terms (and logical terms) are expressible within the system. We may speak of the totality of undefined terms as the *base* of the system, and the formulas expressible in the system are all formulas constructed *on that base*. In general, the totality of formulas constructed on the base of a given system can be divided into three groups: first, all formulas which are provable as theorems within the system; second, all formulas whose negations are provable within the system; and third, all formulas such that neither they nor their negations are provable within the system. For *consistent* systems the first and second groups are *disjoint*, that is, have no formulas in common. Any system whose third group is empty, containing no formulas at all, is said to be *deductively complete*. An alternative way of phrasing this sense of completeness is to say that every formula of the system is such that either it or its negation is provable as a theorem.

Another definition of 'completeness', entailed by but not equivalent to the preceding one, is that a deductive system is complete when every formula constructed on its base is either a theorem or else its addition as an axiom would make the system inconsistent.

An example of an incomplete deductive system would be Euclidean geometry minus the parallel postulate. For the parallel postulate is itself a formula constructible on the base of the Euclidean system, yet neither it nor its negation is deducible from the other postulates. It is clear that although completeness is an important attribute, an incomplete deductive system may be very interesting and valuable. For by investigating the incomplete system of Euclidean geometry without the parallel postulate, we can discover those properties possessed by space independently of the question of whether it is Euclidean or non-Euclidean. Perhaps a more cautious and less misleading formulation of the same point is to say that by investigating the incomplete system we can discover the *common features* of Euclidean and non-Euclidean geometries. Yet for many purposes, a complete system is to be preferred.

6.5 Logistic Systems

Most important of all attributes for a deductive system to possess is that of rigor. A system has rigor when no formula is asserted to be a theorem unless it is logically entailed by the axioms. It is for the sake of rigor that arbitrary rather than familiar symbols are taken as undefined or primitive terms, and the system developed *formally*. Listing clearly all the undefined terms, and explicitly stating all the axioms used as premisses for the theorems, will help to specify precisely which formulas are to be esteemed as theorems and which are not. With the increased emphasis on rigor that characterizes the modern period, critical mathematicians have seen that this is not enough. To achieve rigor, more is required.

A system is rigorous only when its theorems are proved logically, or derived logically from its axioms. It has now been realized that however clearly its axioms are stated, a formal system will lack rigor unless the notion of *logical proof* or *logical derivation* is specified precisely also. All deductive systems of the sort that have been mentioned, even formal deductive systems which contain logical terms in addition to their own special uninterpreted symbols, depend upon 'ordinary logic' for their development. They *assume* logic, in the sense that their theorems are supposed to follow *logically* from their axioms. But they do not specify what this 'logic' is. Hence all earlier deductive systems, for geometry, or physics, or psychology, or the like, contain concealed assumptions which are not explicitly stated. These hidden assumptions are the rules or principles of logic to which one appeals in constructing proofs or derivations of theorems. Hence all those deductive systems fall short of complete rigor, for not all of their presuppositions are acknowledged. Therefore their developments are not entirely rigorous, but more or less loose. The question naturally arises: How can this looseness be eliminated, and greater rigor be achieved? The answer is obvious enough. A deductive system will be developed more rigorously when it is specified not only what axioms are assumed as premisses in deriving the theorems, but also what principles of inference are to be used in the derivations. The axioms must be supplemented by a list of valid argument forms, or principles of valid inference.

The demand for rigor and for system does not stop even here, however. For the sake of rigor, in addition to its own special axioms, a deductive system must specify explicitly what forms of inference are to be accepted as valid. But it would be unsystematic—and probably impossible—simply to list or catalog *all* required rules of logic or valid modes of inference. A deductive system of logic itself must be set up. Such a deductive system will have deduction itself as its subject matter. A system of this type, often referred to as a *logistic system*, must differ from the ordinary, less formal varieties in several important respects. Since its subject matter is deduction itself, the logical terms 'if . . . then . . .', 'and', 'or', 'not', and so on, cannot occur in it with their ordinary meanings simply assumed. In their stead must be uninterpreted symbols. And the logical principles or rules of inference that *it* assumes for the sake of deducing logical theorems from logical axioms must be few in number and explicitly stated.

A second fundamental difference between logistic systems and other formal deductive systems is that in the latter the notion of a significant or 'well formed' formula need not be specified, whereas it is absolutely required in a logistic system. In an ordinary (nonformal) deductive system, it will be obvious which sequences of its words are significant propositions of English (or of whatever the natural language is in which the system is expressed). In a formal but nonlogistical deductive system, the sequences of its symbols are easily divided into those which 'make sense' and those which do not, for they will contain such ordinary logical words as 'if . . . then . . .', 'and', 'or', or 'not', by whose disposition in the sequence it can be recognized as significant or otherwise. An example will make this clear. In a formal deductive system which contains '*A*', '*B*', and '*C*' as uninterpreted primitive symbols, the sequence of symbols 'If any *A* is a *B*, then it is a *C*' is clearly a complete and 'significant' formula which may or may not be provable as a theorem. But the sequence of symbols 'If any *A* is a *B*' is obviously *incomplete*, while the sequence 'And or or *A B* not not if' is clearly nonsense. These are recognized as 'complete' or 'well formed', as 'incomplete' or 'ill formed' by the presence in them of *some* symbols whose meanings are understood. In a logistic system, however, *all* symbols are uninterpreted: there are no familiar words within its formulas (or sequences of symbols) to indicate which are 'well formed' and which are not. Where the symbols '*A*', '*B*', '$\sim$', and '$\supset$' are uninterpreted, there must be some method of distinguishing between a well formed formula like '$A \supset \sim B$' and one like '$AB \supset \sim$', which is not well formed. By our knowledge of the normal interpretations of these symbols we can recognize the difference and classify them correctly, but for the *rigorous* development of our system we must be able to do this in abstraction from the (intended) meanings of the symbols involved.

The matter may be expressed in the following terms. As ordinarily conceived, a nonformal deductive system (interpreted, like Euclidean geometry) is an arrangement or organization of propositions about some specified subject matter. Consisting of propositions, it is a *language* in which the subject matter may be discussed. Understanding the language, we can divide all sequences of its words into those which are meaningful statements and those which are meaningless or nonsensical. This division is effected in terms of meanings and is thus done *nonformally*. In a logistic system the situation is different, for prior to the extrasystematic assignment of meanings or interpretation, *all* sequences of symbols are without meaning. Yet we want, prior to and independent of its interpretation, a comparable division of all of its formulas into two groups. When meanings are assigned to the primitive symbols of a logistic system, some of its formulas will express propositions, while others will not. We may informally characterize a formula which on the intended interpretation becomes a significant statement as a 'well formed formula' (customarily abbreviated '*wff*'). Any formulas which on the intended interpretation do *not* become significant statements are *not* well formed formulas. In a logistic system there must be a *purely formal* criterion for distinguishing well formed formulas from all others. To characterize the criterion as 'purely formal' is to say that it is *syntactical* rather than *semantical*, pertaining to the formal characteristics and arrangements of the symbols in abstraction from their meanings. Thus a logistic system must contain only uninterpreted symbols, and must provide a criterion for dividing sequences of these symbols into two groups, the first of which will contain all well formed formulas, the second containing all others. Of the well formed formulas, some will be designated as Axioms (or Postulates) of the logistic system.

It is also desired to divide all well formed formulas which are not axioms into two groups, those which are theorems and those which are not. The former are those which are derivable from the axioms or postulates, *within the system.* Although uninterpreted, the well formed formulas of a logistic system constitute a 'language' in which derivations or proofs can be set down. Some well formed formulas will be assumed as postulates, and other well formed formulas will be derived from them as theorems. It might be proposed to define 'theorem' as any *wff* which is the conclusion of a valid argument whose premisses include only axioms of the system. This proposed definition of 'theorem' will be acceptable only if the notion of a *valid* argument within the logistic system can be defined formally. Because all *wffs* of the system are uninterpreted, the ordinary notion of validity cannot be used to characterize arguments within the system, for the usual notion of validity is *semantical,* an argument being regarded as valid if and only if the *truth* of its premisses entails the *truth* of its conclusion. Consequently, a purely formal or syntactical *criterion of validity* must be provided for arguments expressed *within the system.* 'Valid' arguments within the system may have not merely postulates or already established theorems as premisses and new theorems as conclusions, but may have as premisses *any wffs,* even those which are neither postulates nor theorems, and as conclusions *wffs* which are not theorems. Of course it is desired that any argument within the system which is syntactically 'valid' will become, on the intended or 'normal' interpretation, a semantically valid argument.

Any logistic system, then, will contain the following elements: (1) a list of primitive symbols which, together with any symbols defined in terms of them, are the only symbols which occur within the system; (2) a purely formal or syntactical criterion for dividing sequences of these symbols into formulas which are well formed (*wffs*) and those which are not; (3) a list of *wffs* assumed as postulates or axioms; (4) a purely formal or syntactical criterion for dividing sequences of well formed formulas into 'valid' and 'invalid' arguments; and (5), derivatively from (3) and (4), a purely formal criterion for distinguishing between theorems and nontheorems of the system.

Different logistic systems may be constructed as systematic theories of different parts of logic. The simplest logistic systems are those which formalize the logic of truth-functional compound statements. These systems are called *propositional calculi* or, less frequently, *sentential calculi.* One particular propositional calculus will be presented and discussed in the following chapter.